THE WICK PRODUCT

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Abstract

We give a survey of the basic properties of the Wick product and its applications.

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§0. Introduction

The Wick product was introduced by G.C. Wick in 1950 in the context of quantum field theory. In 1965 T. Hida and N. Ikeda introduced a closely related concept in probability theory. The concept now plays a crucial role in both mathematical physics and stochastic analysis.

The purpose of this paper is to give an introduction to the Wick product and some of its basic properties. We want to point out some of the different (but related) approaches to the concept and we also hope to clarify the relation between them. Finally we discuss some of the recent applications of the Wick product in stochastic differential equations.

No attempt has been made to be complete in any way, neither with respect to the history of the concept nor with respect to the selection of the associated topics. Indeed, there are so many different aspects of it - and open problems about them - that a complete survey seems impossible at the present time. But we hope this article will convince the reader about the importance of the Wick product and perhaps inspire him or her to consult the literature further and even pursue the concept by own research, for example by solving some of the problems we list in the end!

§1. The Wick product in physics.

In this section we give a brief discussion of the use of Wick product in the theory of quantum fields. Quantum fields represent the synthesis of quantum mechanics with relativity.

Wick products were introduced by Wick [Wi], who called it the "S-product", as a convenient notation to simplify certain complicated expressions in the study of Heissenberg's S-matrix.

We will here concentrate on a simple case, namely that of free Euclidean fields. In this formulation the fields are continued analytically to imaginary time. This changes the Minkowski metric into the technically simpler Euclidean metric and it changes the complex Schrödinger equation into the real heat equation (with a potential), which is more tractable from the point of view of stochastic analysis. The following presentation is based on Simon [S] and Glimm and Jaffe [GJ].

Let $(\Omega, \mathcal{B}, P)$ be a probability space, and fix a random variable U. Consider a formal power series in U and define a formal derivation with respect to U by

$$\frac{\partial}{\partial U}\left(\sum_{n=0}^{\infty} a_n U^n\right) = \sum_{n=0}^{\infty}(n+1)a_{n+1}U^n \tag{1.1}$$

The Wick order of $U^k, : U^k :$, can recursively be defined by

$$\begin{aligned} &: U^0 := 1, \quad \frac{\partial}{\partial U} : U^n := n : U^{n-1} : \\ &\qquad\qquad E[: U^n :] = 0, \quad n = 1, 2, \cdots \end{aligned} \tag{1.2}$$

Hence

$$\begin{aligned} : U : &= U - E(U) \\ : U^2 : &= U^2 - 2UE(U) - E(U^2) + 2(E(U))^2 \\ : U^3 : &= U^3 - 3U^2E(U) - 3UE(U^2) \\ &\quad + 6UE(U)^2 - E(U)^3 + 6E(U)E(U^2) - 6E(U^3). \end{aligned} \tag{1.3}$$

Extend this definition by linearity to convergent power series to obtain e.g.

$$: e^{aU} := E(e^{aU})^{-1} e^{aU} \tag{1.4}$$

We may extend this definition further to more than one random variable as follows:

If $U_1, \cdots, U_n$ are random variables, then we define

$$: U_1^0 \cdots U_N^0 := 1, \tag{1.5}$$

and

$$\frac{\partial}{\partial U_i} : U_1^{n_1} \cdots U_N^{n_N} := n_i : U_1^{n_1} \cdots U_i^{n_i - 1} \cdots U_N^{n_N} :$$
$$E(: U_1^{n_1} \cdots U_N^{n_N} :) = 0, \quad \text{not all } n_i = 0 \tag{1.6}$$

This gives e.g.

$$: UV := UV - UE(V) - VE(U) + 2E(U)E(V) - E(UV) \tag{1.7}$$

Sometimes $: UV :$ is called the Wick product of U and V. Here we will call it the *physical Wick product* to distinguish it from the Wick product we will define later.

The use of the physical Wick product requires extreme care, as most of the usual algebraic properties one is used to in fact no longer hold. We have e.g.

$$: UV : |_{V=U^2} = U^3 - UE(U^2) - U^2E(U) + 2E(U)E(U^2) - E(U^3) \tag{1.8}$$

which in general is different from $: U^3 :$ as given by (1.3)! Furthermore, using (1.7),

$$\begin{aligned} : U1 : &= U1 - UE(1) - 1E(U) + 2E(U)E(1) - E(U1) \\ &= U - U - E(U) + 2E(U) - E(U) = 0 \end{aligned}$$

which clearly is different from $: U :$.

For a zero mean Gaussian random variable U with variance $\sigma^2 = E[U^2]$ we find in particular

$$: U^n := \sigma^n h_n(\frac{U}{\sigma}) \tag{1.9}$$

where h_n is the *n-th Hermite polynomial*, defined by

$$h_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}}); \quad n = 0, 1, 2, \cdots \tag{1.10}$$

so that

$$\begin{aligned} &h_0(x) = 1,\ h_1(x) = x,\ h_2(x) = x^2 - 1,\ h_3(x) = x^3 - 3x, \\ &h_4(x) = x^4 - 6x^2 + 3,\ h_5(x) = x^5 - 10x^3 + 15x, \cdots \end{aligned} \tag{1.11}$$

We will now see how Wick products are used in the quantum field setting:

Let $\mathcal{E}$ be a positive, continuous, nondegenerate bilinear form on the Schwartz space $\mathcal{S} = \mathcal{S}(\mathbf{R}^d)$ of rapidly decreasing smooth functions on $\mathbf{R}^d$ (d being the dimension of space-time). As usual we let $\mathcal{S}' = \mathcal{S}'(\mathbf{R}^d)$ denote the space of tempered distributions on $\mathbf{R}^d$ ($\mathcal{S}'(\mathbf{R}^d)$ is the dual of $\mathcal{S}(\mathbf{R}^d)$). By the Bochner-Minlos theorem [GV], there exists a measure $\mu_{\mathcal{E}}$ on $\mathcal{S}'$ such that

$$T(\phi) := \int_{\mathcal{S}'} e^{i<\omega,\phi>} d\mu_{\mathcal{E}}(\omega) = e^{-\frac{1}{2}\mathcal{E}(\phi,\phi)}; \phi \in \mathcal{S} \tag{1.12}$$

where $< \omega, \phi >= \omega(\phi)$ is the action of $\omega \in \mathcal{S}'$ on $\phi \in \mathcal{S}$.

Let $\mathbf{G} = L^2(\mathbf{R}^d, \mathcal{E})$. We can form the symmetric tensor product $\mathcal{G}_n$ of n copies of $\mathbf{G}$ and define the direct sum

$$\mathcal{G} = \bigoplus_{n=0}^{\infty} \mathcal{G}_n \tag{1.13}$$

which is called *the Fock space* associated with $\mathbf{G}$ and which is unitarily equivalent to $L^2(\mathcal{S}'(\mathbf{R}^d), \mu_{\mathcal{E}})$. The space $\mathcal{G}_n$ is spanned by the functions

$$\omega \to :< \omega, \phi_1 > \cdots < \omega, \phi_n >: \quad (\phi_i \in \mathcal{S})$$

and we have in fact that $:< \omega, \phi_1 > \cdots < \omega, \phi_n >:$ is the orthogonal projection of $< \omega, \phi_1 > \cdots < \omega, \phi_n >$ onto $\mathcal{G}_n$.

Quantum fields are described by probability measures on $\mathcal{S}'(\mathbf{R}^d)$ that satisfy the *Osterwalder-Schrader axioms.* Specifically

(OS 0) $T(\phi)$ should be *entire* (analytic) in the sense that the function

$$(z_1, \cdots, z_n) \to T(\sum_{i=1}^{n} z_i\phi_i) \quad z_i \in \mathbf{C}$$

is analytic in $\mathbf{C}^n$, for all n.

(OS 1) $|T(\phi)| \leq \exp[c(\|\phi\|_1 + \|\phi\|_p^p)]$ for $p \in [1, 2]$

(OS 2) $T(\phi)$ is invariant under Euclidean symmetries in $\mathbf{R}^d$

(OS 3) $T(\phi)$ should satisfy reflection positivity, i.e. the matrix

$$[T(\phi_i - \theta\phi_j)]_{i,j=1}^{n}$$

should be positive definite (θ is time reflection)

(OS 4) The fields should be ergodic in the sense that the time translation subgroup acts ergodically on the measure space $(\mathcal{S}'(\mathbf{R}^d), \mu_{\mathcal{E}})$.

Each axiom corresponds to a physical property that is not important in our context. In the case of a *free field* we choose

$$\mathcal{E}(\phi, \psi) = (\phi, (-\Delta_d + m^2)^{-1}\psi)_{L^2(\mathbf{R}^d)}$$

where $\Delta_d = \frac{1}{2}\sum_{j=1}^{d}\frac{\partial^2}{\partial x_j}$ is the Laplacian in $\mathbf{R}^d$ and $m > 0$ represents the mass. One can then prove that the free field satisfies the Osterwalder-Schrader axioms above. One may consider the free field as infinitely many uncoupled harmonic oscillators. Often interacting fields are constructed by perturbing the free fields. In the interpretation of free fields the space $\mathcal{G}_n$ is called the space of n particle vectors and in particular $\mathcal{G}_0$ corresponds to the vacuum state.

§2. The Wick product in stochastic analysis.

In stochastic analysis the Wick product was first introduced by T. Hida and N. Ikeda in 1965 [HI]. A systematic, general account of the traditions of both mathematical physics and probability theory regarding this subject was given by Dobrushin and Minlos in 1977 [DM]. In 1989 P.A. Meyer and J.A. Yan [MY] defined the Wick product of two *Hida distributions* (or *white noise functionals*, see §2 c) below) in terms of their *$\mathcal{S}$-transforms* (see §4). Motivated by the interpretation and solution of certain stochastic (ordinary and partial) differential equations, the Wick product in the $L^2(\mu)$- and the $L^1(\mu)$-setting was introduced and applied in [LØU 1], [LØU 2] and [HLØUZ]. (Here μ is the white noise probability measure defined in §2 a) below).

Today the Wick product is important in the study of stochastic (ordinary and partial) differential equations. In general one can say that the use of this product corresponds to - and extends naturally - the use of Ito integrals. We now explain this in more detail.

2a) THE WHITE NOISE PROBABILITY SPACE

Two fundamental concepts in stochastic analysis are *white noise* and *Brownian motion*. Let us therefore start by recalling some of the basic definitions and features of *the white noise probability space* (Here we give only the most basic results. For a complete account we refer to [HKPS]):

Referring to the construction in (1.12), let $\mu = \mu_I$ be the measure on $\mathcal{S}' = \mathcal{S}'(\mathbf{R}^d)$ corresponding to the bilinear form

$$\mathcal{E}_I(\phi, \psi) = (\phi, \psi)_{L^2(\mathbf{R}^d)} = \int_{\mathbf{R}^d} \phi\psi dx \quad ; \phi, \psi \in \mathcal{S}$$

In other words, μ is defined by the property

(2.1) $$\int_{\mathcal{S}'(\mathbf{R}^d)} e^{i<\omega,\phi>} d\mu(\omega) = e^{-\frac{1}{2}\|\phi\|^2} \quad \phi \in \mathcal{S}(\mathbf{R}^d)$$

where $\|\phi\|^2 = (\phi, \phi)_{L^2(\mathbf{R}^d)}$.

Let $\mathcal{B}$ denote the Borel sets on $\mathcal{S}'$ (equipped with the weak star topology). Then the triple $(\mathcal{S}'(\mathbf{R}^d), \mathcal{B}, \mu)$ is called *the white noise probability space.*

DEFINITION 2.1 The *white noise process* is a map

$$W : \mathcal{S} \times \mathcal{S}' \to \mathbf{R}$$

given by

(2.2) $$W(\phi, \omega) = W_\phi(\omega) = < \omega, \phi > \quad \omega \in \mathcal{S}', \phi \in \mathcal{S}$$

It is not difficult to prove that if $\phi \in L^2(\mathbf{R}^d)$ and we choose $\phi_n \in \mathcal{S}$ such that $\phi_n \to \phi$ in $L^2(\mathbf{R}^d)$ then

$$(2.3) \qquad < \omega, \phi > := \lim_{n\to\infty} < \omega, \phi_n > \quad (\text{limit in } L^2(\mu))$$

exists and is independent of the choice of $\{\phi_n\}$. In particular, if we define

$$(2.4) \qquad \tilde{B}_x(\omega) := \tilde{B}_{x_1,\cdots,x_d}(\omega) := < \omega, \chi_{[0,x_1]\times\cdots\times[0,x_d]}(\cdot) >$$

then $\tilde{B}_x$ has an *x-continuous version* B_x which then becomes a *d-parameter Brownian motion.*

The *d-parameter Wiener-Ito integral* of $\phi \in L^2(\mathbf{R}^d)$ is then defined by

$$(2.5) \qquad \int_{\mathbf{R}^d} \phi(y)dB_y(\omega) = < \omega, \phi >$$

Combining (2.2) and (2.5) and using integration by parts for Wiener-Ito integrals we see that white noise may be regarded as the distributional derivative of Brownian motion:

$$(2.6) \qquad W_{x_1,\cdots,x_d} = \frac{\partial^d B_{x_1,\cdots,x_d}}{\partial x_1 \partial x_2 \cdots \partial x_d}$$

(For more details see e.g. [LØU1] or [HLØUZ]).

Figure 1 shows computer simulations of the white noise process $W_{\phi_x}(\omega)$, where $\phi(y) = \chi_{[0,h]\times[0,h]}(y); y \in \mathbf{R}^2$ and $\phi_x(y) = \phi(y - x)$ is the x-shift of ϕ $(x \in \mathbf{R}^2)$.

Figure 1
Two sample paths of white noise. ($h = \frac{1}{50}, h = \frac{1}{20}$)

2b) THE WIENER-ITO CHAOS EXPANSION

Of special interest will now be the space $L^2(\mathcal{S}'(\mathbf{R}^d), \mu)$ or $L^2(\mu)$ for short. The celebrated *Wiener-Ito chaos expansion theorem* says that every $f \in L^2(\mu)$ has the form

$$(2.7) \qquad f(\omega) = \sum_{n=0}^{\infty} \int_{(\mathbf{R}^d)^n} f_n(u) dB_u^{\otimes n}$$

where $f_n \in \hat{L}^2(\mathbf{R}^{nd})$, i.e. f_n is *symmetric* in its nd variables (in the sense that $f_n(u_{\sigma_1}, \cdots, u_{\sigma_{nd}}) = f_n(u_1, \cdots, u_{nd})$ for all permutations σ of $(1, 2, \cdots, nd)$) and f_n is square integrable with respect to Lebesgue measure on $\mathbf{R}^{nd}$. The terms on the right hand side of (2.7) are the *multiple Ito integrals*, defined in [I]. (If $d = 1$ they are just iterated Ito integrals on the subspace $u_1 < u_2 < \cdots < u_n$ and multiplied by $n!$)

With f, f_n as in (2.7) we have

$$\|f\|^2_{L^2(\mu)} = \sum_{n=0}^{\infty} n! \|f_n\|^2_{L^2(\mathbf{R}^{nd})} \tag{2.8}$$

There is an equivalent expension of $f \in L^2(\mu)$ in terms of the Hermite polynomials h_n defined in (1.10) above. We now explain this more closely:

For $n = 1, 2, \cdots$ let $\xi_n(x)$ be the Hermite *function* of order n, i.e.

$$\xi_n(x) = \pi^{-\frac{1}{4}}((n-1)!)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} h_{n-1}(\sqrt{2}x), \quad x \in \mathbf{R} \tag{2.9}$$

where h_n is defined in (1.10).

Then ξ_n is an eigenfunction with eigenvalue $2n$ of the operator

$$A = -(\frac{d}{dx})^2 + x^2 + 1.$$

Moreover, $\{\xi_n\}_{n=1}^{\infty}$ forms an orthonormal basis for $L^2(\mathbf{R})$. Therefore the family $\{e_\alpha\}$ of tensor products

$$e_\alpha := e_{\alpha_1, \cdots, \alpha_d} := \xi_{\alpha_1} \otimes \cdots \otimes \xi_{\alpha_d} \tag{2.10}$$

(where α denotes the multi-index $(\alpha_1, \cdots, \alpha_d)$) forms an orthonormal basis for $L^2(\mathbf{R}^d)$.

This is the basis we will use in the rest of this paper. Note that $e_\alpha \in \mathcal{S}(\mathbf{R}^d)$ for all $\alpha = (\alpha_1, \cdots, \alpha_d)$. Now assume that the family of all multi-indeces $\beta = (\beta_1, \cdots, \beta_d)$ is given a fixed ordering

$$(\beta^{(1)}, \beta^{(2)}, \beta^{(3)}, \cdots, \beta^{(n)}, \cdots), \quad \text{where } \beta^{(k)} = (\beta_1^{(k)}, \cdots, \beta_d^{(k)}),$$

and put

$$e_n = e_{\beta^{(n)}}; \ n = 1, 2, \cdots$$

Let $\alpha = (\alpha_1, \cdots, \alpha_m)$ be a multi-index. Then it was shown by Ito [I] that

$$\int_{(\mathbf{R}^d)^n} e_1^{\hat{\otimes}\alpha_1} \hat{\otimes} e_2^{\hat{\otimes}\alpha_2} \hat{\otimes} \cdots \hat{\otimes} e_m^{\hat{\otimes}\alpha_m} dB^{\hat{\otimes}n} = \prod_{j=1}^{m} h_{\alpha_j}(\theta_j) \tag{2.11}$$

where $\theta_j(\theta) = \int\limits_{\mathbf{R}^d} e_j(x)dB_x(\omega), n = |\alpha_1| + \cdots + |\alpha_m|$ and $\hat{\otimes}$ denotes the *symmetrized tensor product*, so that, e.g., $f\hat{\otimes}g(x_1, x_2) = \frac{1}{2}[f(x_1)g(x_2) + f(x_2)g(x_1)]$ if $x_i \in \mathbf{R}$ and similarly for more than two variables.

If we define, for each multiindex $\alpha = (\alpha_1, \cdots, \alpha_m)$,

$$H_\alpha(\omega) = \prod_{j=1}^{m} h_{\alpha_j}(\theta_j) \tag{2.12}$$

then we see that (2.11) can be written

$$\int\limits_{(\mathbf{R}^d)^n} e^{\hat{\otimes}\alpha} dB^{\otimes|\alpha|} = H_\alpha(\omega) \tag{2.13}$$

using multiindex notation: $e^{\hat{\otimes}\alpha} = e_1^{\hat{\otimes}\alpha_1} \otimes \cdots \otimes e_m^{\hat{\otimes}\alpha_m}$ if $e = (e_1, e_2, \cdots)$. Since the family $\{e^{\hat{\otimes}\alpha}; |\alpha| = n\}$ forms an orthonormal basis for $\hat{L}^2((\mathbf{R}^d)^n)$, we see by combining (2.7) and (2.13) that we also have the representation

$$f(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega) \tag{2.14}$$

(the sum being taken over all multi-indeces α of non-negative integers). Moreover, it can be proved that

$$\|f\|^2_{L^2(\mu)} = \sum_\alpha \alpha! c_\alpha^2 \tag{2.15}$$

where $\alpha! = \alpha_1!\alpha_2!\cdots\alpha_m!$ if $\alpha = (\alpha_1, \cdots, \alpha_m)$.

We refer to [HKPS] for more information.

2c) THE HIDA TEST FUNCTION SPACE $(\mathcal{S})$ AND THE HIDA DISTRIBUTION SPACE $(\mathcal{S})^*$.

There is a subspace of $L^2(\mu)$ which in some sense corresponds to the Schwartz subspace $\mathcal{S}(\mathbf{R}^d)$ of $L^2(\mathbf{R}^d)$. This space is called the *Hida test function space* and is denoted by $(\mathcal{S})$. Using the recent characterization due to one of us (see [Z1]), a simple description of $(\mathcal{S})$ can be given as follows:

DEFINITION 2.2 Let $f \in L^2(\mu)$ have the chaos expansion

$$f(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega) \tag{2.16}$$

Then f is a *Hida test function*, i.e. $f \in (\mathcal{S})$, if

$$A_f(k) := \sup_\alpha c_\alpha^2 \alpha!(2\mathrm{N})^{\alpha k} < \infty \quad \text{for all } k < \infty \tag{2.17}$$

where

$$(2\mathrm{N})^\alpha := \prod_{j=1}^{m} (2^d \beta_1^{(j)} \cdots \beta_d^{(j)})^{\alpha_j} \quad \text{if} \quad \alpha = (\alpha_1, \cdots, \alpha_m). \tag{2.18}$$

The topology of $(\mathcal{S})$ can be described by the metric given by

$$d(0, f) = \sum_{k=1}^{\infty} 2^{-k} \frac{A_f(k)}{1 + A_f(k)} \quad \text{for} \quad f \in (\mathcal{S}) \tag{2.19}$$

and

$$d(f_1, f_2) = d(0, f_1 - f_2) \quad \text{for} \quad f_1, f_2 \in (\mathcal{S}). \tag{2.20}$$

The *Hida distribution space* $(\mathcal{S})^*$ is just the dual of $(\mathcal{S})$. By [Z1] we may describe this space as follows:

THEOREM 2.3 A *Hida distribution* F is a formal series

$$F = \sum_\alpha b_\alpha H_\alpha \tag{2.21}$$

where

$$\sup_\alpha b_\alpha^2 \alpha!((2\mathrm{N})^{-\alpha})^q < \infty \quad \text{for some} \quad q > 0 \tag{2.22}$$

If $F \in (\mathcal{S})^*$ is given by (2.21) and $f \in (\mathcal{S})$ is given by (2.14), the action of F on f is given by

$$< F, f >= \sum_\alpha \alpha! b_\alpha c_\alpha \tag{2.23}$$

Note that no assumptions are made regarding the convergence of the formal series in (2.21).

We can in a natural way regard $L^2(\mu)$ as a subspace of $(\mathcal{S})^*$. In particular, if $X \in L^2(\mu)$ then by (2.23) the action of X on $f \in (\mathcal{S})$ is given by

$$< X, f >= E[X \cdot f]$$

On the other hand it is easy to see that $(\mathcal{S}) \subset L^p(\mu)$ for all $p < \infty$ and the inclusion map is continuous. We conclude that

$$(\mathcal{S}) \subset L^p(\mu) \subset L^q(\mu) \subset (\mathcal{S})^* \quad \text{if} \quad 1 < q \leq p < \infty \tag{2.24}$$

However, it is *not* the case that $L^1(\mu) \subset (\mathcal{S})^*$ (see [HLØUZ]).

EXAMPLE. The discussion below works for all d, but for simplicity let us assume that $d = 1$.

We have already introduced the white noise process $W_\phi(\omega)$ as a map from $\mathcal{S} \times \mathcal{S}'$ into $\mathbf{R}$ (see (2.2)). It is also possible to define a *pointwise* version $W_t(\omega); t \in \mathbf{R}$, but then as an element of $(\mathcal{S})^*$, as follows: Define

$$W_t(\omega) = \sum_{k=1}^{\infty} e_k(t) H_{\epsilon_k}(\omega) = \sum_{k=1}^{\infty} e_k(t) h(\theta_k) \tag{2.25}$$

where $\epsilon_k = (0, \cdots, 1)$ with 1 on $k'th$ place, $k = 1, 2, \cdots$. Then by (2.18)

$$(2\mathrm{N})^{\epsilon_k} = 2k$$

and, using the notation of (2.22) with $\alpha = \epsilon_k$,

$$\sup_\alpha b_\alpha^2 \alpha! (2\mathrm{N})^{-\alpha q} = \sup_k e_k^2(t) \cdot 1 \cdot (2k)^{-q} < \infty$$

for some $q > 0$.

This shows that *the pointwise version W_t of white noise* exists as an element of $(\mathcal{S})^*$. Since, for $\phi \in \mathcal{S}(\mathbf{R}^d)$,

$$W_\phi(\omega) = < \omega, \phi > = \sum_k (\phi, e_k) < \omega, e_k > = \sum_k (\phi, e_k) H_{\epsilon_k}(\omega)$$

we see that we may regard $W_t(\omega)$ as the limit in $(\mathcal{S})^*$ of $W_{\phi_n}(\omega)$ as $\phi_n \longrightarrow \delta_t$, the point mass at t. Or, heuristically,

$$W_t(\omega) = \int_{\mathbf{R}} \delta_t(u) dB_u(\omega) \tag{2.26}$$

2d) THE WICK PRODUCT

We are now ready to give the definition of *the Wick product* $F \diamond G$ of two Hida distributions F, G:

DEFINITION 2.4 Let $F = \sum_\alpha a_\alpha H_\alpha, G = \sum_\beta b_\beta H_\beta$ be two elements of $(\mathcal{S})^*$. Then *the Wick product* of F and G is the element $F \diamond G$ in $(\mathcal{S})^*$ given by

$$F \diamond G = \sum_{\alpha, \beta} a_\beta b_\beta H_{\alpha+\beta} = \sum_\gamma c_\gamma H_\gamma \tag{2.27}$$

where $c_\gamma = \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta$.

REMARK. It follows from the characterization (2.19) that the Wick product of two elements of $(\mathcal{S})^*$ again is an element of $(\mathcal{S})^*$. Similarly, from the characterization (2.15) we see that

$$f, g \in (\mathcal{S}) \Rightarrow f \diamond g \in (\mathcal{S})$$

If

$$f = \sum_{n=0}^{\infty} \int f_n dB^{\otimes n} \quad \text{and} \quad g = \sum_{m=0}^{\infty} \int g_m dB^{\otimes m} \tag{2.28}$$

are two functions in $L^2(\mu)$, their Wick product can be expressed by

$$f \diamond g = \sum_{n,m=0}^{\infty} \int f_n \hat{\otimes} g_m dB^{\otimes(n+m)} \tag{2.29}$$

whenever the sum converges (in $L^1(\mu)$).

Equivalently, if we use the Hermite expansions

$$f = \sum_{\alpha} a_\alpha H_\alpha \quad \text{and} \quad g = \sum_{\beta} b_\beta H_\beta \tag{2.30}$$

then the Wick product is given by

$$f \diamond g = \sum_{\alpha,\beta} a_\alpha b_\beta H_{\alpha+\beta} = \sum_{\gamma} c_\gamma H_\gamma, \tag{2.31}$$

where $c_\gamma = \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta$.

In particular, if we choose $d = 1$ and f Gaussian, i.e.

$$f(\omega) = \int_{\mathbf{R}} f_1(x) dB_x(\omega), \quad \text{where} \quad f_1 \in L^2(\mathbf{R}) \quad \text{is deterministic} \tag{2.32}$$

then one obtains from (2.31) and (1.9) that

$$f^{\diamond n} = \|f_1\|^n h_n\left(\frac{f}{\|f_1\|}\right) =: f^n : \tag{2.33}$$

Thus the physical Wick power and the (stochastic analysis) Wick power coincide in this case. However, for other products they do *not* coincide. For example, if

$$U = \int \chi_{[0,t]}(s) dB_s = B_t \quad \text{and} \quad V = U^{\diamond 2}$$

we have by (2.24) and (2.28)

$$U \diamond V = U \diamond (U^{\diamond 2}) = U^{\diamond 3} = t^{3/2} h_3(\frac{B_t}{\sqrt{t}}) = B_t^3 - 3tB_t$$

while, by (1.8),

$$: UV := B_t^3 - tB_t$$

In fact, the physical Wick product $: UV :$ should not be called a *product*, because as an binary operation it is not even associative. On the other hand it is trivial to verify that the (stochastic analysis) Wick product $U \diamond V$ is associative, as well as commutative and distributive over addition. So, considered as a binary operation on $(\mathcal{S})$ or $(\mathcal{S})^*$ it forms a ring with unit 1 (the constant function).

2e) APPLICATIONS

In quantum physics the physical Wick product may be regarded as a renormalized product, introduced to avoid selfinteractions. In stochastic analysis the Wick product is natural because it is implicit in the Ito integral (and, more generally, in the Skorohod integral if the integrand is not adapted). More precisely, if $Y_t(\omega)$ is a (suitable) stochastic process adapted to the filtration $\mathcal{F}_t$ generated by the (1-parameter) Brownian motion B_t then we have

$$\int_{\mathbf{R}} Y_t(\omega) dB_t(\omega) = \int_{\mathbf{R}} Y_t \diamond W_t dt \tag{2.34}$$

Here $W_t \in (\mathcal{S})^*$ is the pointwise white noise defined in (2.25) and the integral on the right is to be regarded as an integral in $(\mathcal{S})^*$. If one prefers to work with less singular objects one can reformulate (2.34) as follows:

$$\int_{\mathbf{R}} (\phi * Y)_t \delta B_t = \int_{\mathbf{R}} Y_t \diamond W_{\phi_t} dt \quad ; \ \phi \in \mathcal{S}(\mathbf{R}) \tag{2.35}$$

where $*$ denotes convolution with respect to Lebesgue measure on $\mathbf{R}$, i.e.

$$(\phi * Y)_t(\omega) = \int_{\mathbf{R}} \phi(s) Y_{s-t}(\omega) ds \tag{2.36}$$

and

$$\phi_t(s) = \phi(s-t) \quad \text{is the t-shift of} \quad \phi. \tag{2.37}$$

As before W_ϕ is the white noise "smeared out by ϕ" given by (2.2). The notation "δB_t" indicates that the integral should be interpreted as a *Skorohod* integral, since the integrand in (2.35) need not be adapted even if Y_t is. In fact, (2.35) remains true for nonadapted processes Y_t. (The Skorohod integral coincides with the Ito integral if the integrand is adapted. See e.g. [ZN] for more information). There are several versions of (2.34), see e.g. [LØU 2] and [AP]. The "smeared out" version (2.35) is proved in [ØZ].

From (2.34) and (2.35) we conclude that if we interpret a white noise differential equation

$$\frac{dX_t}{dt} = b(X_t) + \sigma(X_t) \cdot W_t \quad ; \; X_0 = x \in \mathbf{R}^n \tag{2.38}$$

as an Ito integral equation

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s, \tag{2.39}$$

we are really saying that the product on the right hand side of (2.38) is interpreted as a Wick product:

$$\frac{dX_t}{dt} = b(X_t) + \sigma(X_t) \diamond W_t \quad ; \; X_0 = x \tag{2.40}$$

Moreover, if all products in the equation are Wick products, then the equation can be solved by *ordinary* (deterministic) calculus rules as long as all the products are taken in the Wick sense. This can facilitate the solution of stochastic differential equations considerably. We illustrate this by some examples:

EXAMPLE 2.5. Consider the equation for, say, a population growth in a "white noise environment":

$$\frac{dX_t}{dt} = X_t W_t \quad ; \; X_0 \quad \text{given} \tag{2.41}$$

or, in Ito differential notation,

$$dX_t = X_t dB_t \quad ; \; X_0 \quad \text{given.} \tag{2.42}$$

Let us first assume that $X_0 = x$ (constant). Rather than using Ito calculus on (2.42) we use Wick calculus on the Wick interpretation of (2.41), i.e.

$$\frac{dX_t}{dt} = X_t \diamond W_t \quad ; \; X_0 = x \tag{2.43}$$

Based on the analogy from deterministic calculus we guess that the solution of (2.43) is

$$X_t = x \cdot \mathrm{Exp}\Big(\int_0^t W_s ds\Big) = x \cdot \mathrm{Exp} B_t, \tag{2.44}$$

where

$$\text{(2.45)} \qquad \mathrm{Exp} B_t = \sum_{n=0}^{\infty} \frac{1}{n!} B_t^{\diamond n}$$

is the *Wick exponential* of B_t. (See Example 2.7 for a more general discussion of this.)

Indeed, we verify that (2.44) is a solution by verifying (using (2.45)) that

$$\frac{d}{dt}(\mathrm{Exp}(\int_0^t W_s ds)) = \mathrm{Exp}(\int_0^t W_s ds) \diamond W_t.$$

On the other hand, it is well known that Ito calculus applied to (2.42) gives the solution

$$\text{(2.46)} \qquad X_t = x \cdot \exp(B_t - \frac{1}{2}t)$$

The two solutions (2.44) and (2.46) are indeed the same, as can be seen by combining (2.33) with the following well known equation for Hermite polynomials:

$$\text{(2.47)} \qquad \sum_{n=0}^{\infty} h_n(x) \frac{t^n}{n!} = e^{tx - \frac{1}{2}t^2}.$$

Now suppose that the initial value X_0 is not a constant x but a random variable which is *anticipating*, i.e. depending on $\mathcal{F}_t$ for some $t > 0$. Then X_s will not be adapted and the Ito differential in (2.42) is no longer well-defined. However, the Wick product interpretation (2.43) makes sense as before and Wick calculus now leads to the solution

$$\text{(2.48)} \qquad X_t = X_0 \diamond \mathrm{Exp}\ B_t$$

(If $X_0 = x$ (constant), then (2.48) reduces to (2.44), since the Wick product reduces to the ordinary product if one of the factors are deterministic). Using (2.35) we see that the process X_t given by (2.48) solves the *Skorohod* integral equation

$$\text{(2.49)} \qquad X_t = X_0 + \int_0^t X_s \delta B_s \quad ; \ t \geq 0.$$

This is an extra advantage with the Wick calculus: It works equally well for adapted and nonadapted processes.

It is interesting to compare the Wick solution (2.48) with a solution obtained by completely different methods by Buckdahn [B]. Specializing his general solution formula to our case we get the solution

$$\text{(2.50)} \qquad X_t(\omega) = X_0(A_t\omega) \cdot \exp(B_t - \frac{1}{2}t)$$

where A_t is the shift operator on $C[0,\infty)$ given by

$$(A_t\omega)(s) = \omega(s) - t \wedge s \quad ; \; s \geq 0, \omega \in C[0,\infty)$$

By uniqueness of equation (2.41) we know that the two solutions (2.48) and (2.50) must be the same, but this is not easy to see directly!

EXAMPLE 2.6 (The stochastic Volterra equation)

The classical deterministic Volterra equation (of the second kind) has the form

$$X_t = Y_t + \int_0^t \gamma(t,s) X_s ds \quad ; \; t \geq 0 \tag{2.51}$$

where $\gamma(t,s), Y_t$ are given functions. If the system is randomly perturbed or if there is insufficient information about the function $\gamma(t,s)$ a natural mathematical formulation would be to put

$$\gamma(t,s) = b(t,s) + \sigma(t,s) W_s \tag{2.52}$$

which would lead to the *stochastic Volterra equation*

$$X_t = Y_t + \int_0^t b(t,s) X_s ds + \int_0^t \sigma(t,s) X_s \cdot W_s ds \tag{2.53}$$

Again there is a question what we should mean by the product $\sigma(t,s)X_s \cdot W_s$ on the right hand side of (2.53). In the general case we allow Y_t to be a random variable, possibly anticipating. And then - following the discussion in the previous example - it is natural to interpret (2.53) in the Skorohod sense, i.e.

$$X_t = Y_t + \int_0^t b(t,s) X_s ds + \int_0^t \sigma(t,s) X_s \delta B_s, \tag{2.54}$$

since we cannot expect X_t to be adapted in this case.

It is proved in [ØZ] that under suitable conditions the solution X_t of (2.54) is given by

$$X_t = Y_t + \int_0^t H(t,s) \diamond Y_s ds, \tag{2.55}$$

where

$$H(t,s) = H(t,s,\omega) = \sum_{n=1}^{\infty} K_n(t,s);$$

with

$$K_1(t,s) = K_1(t,s,\omega) = b(t,s) + \sigma(t,s)W_s$$

and inductively

$$K_{n+1}(t,s) = \int_0^t K_1(t,u) \diamond K_n(u,s)du.$$

In particular, if $Y_s(\omega) = Y(\omega)$ (i.e. constant with respect to s), we get a striking, and perhaps surprising, connection between this general solution X_t and the solution x_t in the special case when $Y_t \equiv 1$, i.e.

$$x_t = 1 + \int_0^t b(t,s)x_s ds + \int_0^t \sigma(t,s)x_s dB_s.$$

By (2.55) we see that the connection is

$$X_t = Y \diamond x_t$$

For details we refer to [ØZ].

EXAMPLE 2.7 (Fluid flow in porous media)

The pressure equation for the flow of an incompressible fluid in a porous medium is

$$\begin{cases} \operatorname{div}(K \cdot \nabla p) = -f & \text{in } D \\ p = 0 & \text{on } \partial D \end{cases} \tag{2.56}$$

where (at a fixed instant of time t) D is a given domain in $\mathbf{R}^d (d = 3)$, f is the given source rate for the fluid in D and $K = K(x)$ is the *permeability* of the medium at the point x. In a typical porous rock the values of $K(x)$ are changing rapidly with x and it is hopeless to find an analytic description of this function. Therefore it is natural to use a stochastic approach, where we represent the function $K(x)$ by a *positive noise* $K(x,\omega)$. Measurements of porous rock samples indicate that such a positive noise should have - at least approximately - the following properties:

(i) $x \neq y \Rightarrow K(x,\cdot)$ and $K(y,\cdot)$ are independent

(ii) $\{K(x,\cdot)\}_{x\in\mathbf{R}^d}$ is a stationary process, i.e. for all $x_1,\cdots,x_n \in \mathbf{R}^d$ the distribution of $(K(x_1+h,\cdot), K(x_2+h,\cdot),\cdots,K(x_n+h,\cdot))$ is independent of $h \in \mathbf{R}^d$

(iii) $K(x,\cdot)$ has a lognormal distribution, for all $x \in \mathbf{R}^d$

There does not exist an ordinary stochastic process with these properties. However, if we consider *distribution valued* (or generalized) stochastic processes $K(\phi,\omega)$ and interpret (i) to mean

$(i)'$ $\quad \operatorname{supp}\phi \cap \operatorname{supp}\psi = \emptyset \Rightarrow K(\phi,\cdot)$ and $K(\psi,\cdot)$ are independent then a good candidate for such a positive noise is the *Wick exponential of white noise.* In other words, we put

$$K(x,\omega) := \operatorname{Exp} W_{\phi_x}(\omega) := \sum_{n=0}^{\infty} \frac{1}{n!} W_{\phi_x}^{\diamond n}(\omega) \tag{2.57}$$

where as before $\phi_x(y) = \phi(y - x)$ is the x-shift of the test function $\phi \in \mathcal{S}(\mathbf{R}^3)$.

Combining (2.11) and (2.47) we see that

$$\operatorname{Exp} W_\phi = \exp(W_\phi - \frac{1}{2}\|\phi\|^2); \phi \in \mathcal{S}(\mathbf{R}^d)$$

Moreover, the series (2.57) converges in $L^p(\mu)$ for all $p < \infty$. This follows from the Carlen-Kree estimate [CK] (see Theorem 5.10).

Computer simulations of 2-parameter (i.e. $d = 2$) positive noise $K(x, \omega)$ are shown on Figure 2.

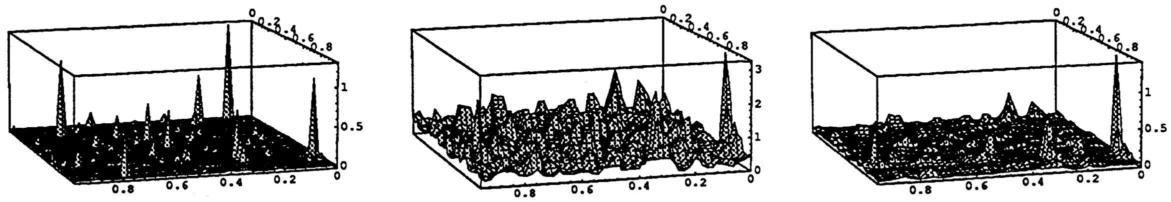


Figure 2
Three sample paths of positive noise $Exp[\epsilon W_\phi]$.
$(h = \frac{1}{50}, \epsilon = 0.05)$ $(h = \frac{1}{20}, \epsilon = 0.05)$ $(h = \frac{1}{20}, \epsilon = 0.1)$

At this point it is necessary to make precise what we mean by a *solution* p of (2.56). It has been known for a long time (see [Wa]) that in higher dimensions stochastic partial differential equations only have solutions in the (Schwartz) distribution sense, i.e. solutions which are distribution valued stochastic processes. So we should look for solutions $p = p(\phi, \omega); \phi \in \mathcal{S}(\mathbf{R}^d), \omega \in \mathcal{S}(\mathbf{R}^d)$. If we fix the test function ϕ and consider its shifts $\phi_x(\cdot); x \in \mathbf{R}^d$, we get the process

$$X(\phi, x, \omega) = p(\phi_x, \omega) \tag{2.58}$$

Now note that taking distributional derivatives of p with respect to ϕ is (by integration by parts) the same as taking derivatives of X with respect to x. For example, if $d = 1, F \in \mathcal{S}'(\mathbf{R})$ and D denotes the differentiation operator we have

$$< DF, \phi_x(\cdot) >= - < F, D\phi_x(\cdot) >= - < F, \frac{\partial}{\partial y}\phi(y - x) >=< F, \frac{d}{dx}\phi_x(y) > .$$

This leads to the following definition:

DEFINITION 2.8. Let $p \geq 1$. *An L^p functional process* is a function

$$X : \mathcal{S}(\mathbf{R}^d) \times \mathbf{R}^d \times \mathcal{S}'(\mathbf{R}^d) \to \mathbf{R}$$

such that

(i) $x \to X(\phi, x, \omega)$ is (Borel) measurable for all $\phi \in \mathcal{S}, \omega \in \mathcal{S}'$

and

(ii) $\omega \to X(\phi, x, \omega)$ belongs to $L^p(\mu)$ for all $\phi \in \mathcal{S}, x \in \mathbf{R}^d$

The heuristic interpretation of X is that $X(\phi, x, \omega)$ is the result we get if we measure X using the test function (or "window") ϕ shifted to the point x and in the "experiment" ω.

Note that the positive noise K given by (2.57) is an L^2 functional process (in fact L^p for all $p < \infty$) according to this definition.

If K in (2.56) is represented by (2.57), the pressure p is represented by an L^p functional process $X(\phi, x, \omega)$ and the product "$K \cdot \bigtriangledown p$" is interpreted as the Wick product $K \diamond \bigtriangledown X$, then equation (2.56) becomes

$$\begin{cases} \operatorname{div}(K(\phi, x) \diamond \bigtriangledown X(\phi, x)) = -(f * \phi)(x) & ; x \in D \\ X(\phi, x) = 0 & ; x \in \partial D \end{cases} \tag{2.59}$$

where $(f * \phi)(x) = \int\limits_{\mathbf{R}^d} f(y)\phi(y - x)dy$ is the convolution and the derivatives (div, grad) are taken with respect to x.

This equation seems suitable for a Wick calculus approach. However, in this case we encounter a new difficulty: It turns out that even in the 1-dimensional case ($d = 1$) there does not exist an L^p functional process solution of (2.59) for $p > 1$. In fact, this equation does not even have solution in $(\mathcal{S})^*$. (See §4). So in order to handle such equations it is necessary to extend the definition of the Wick product beyond $L^2(\mu)$ - and even beyond $(\mathcal{S})^*$. In [HLØUZ] the Wick product is (partially) extended to $L^1(\mu)$ and this is sufficient to solve, for example, the stochastic Schrödinger equation with a positive noise potential. The extension goes as follows:

DEFINITION 2.9. Let $X, Y \in L^1(\mu)$ and suppose there exist $X_n, Y_n \in L^2(\mu)$ such that

$$X_n \to X \quad \text{in } L^1(\mu), Y_n \to Y \quad \text{in } L^1(\mu) \tag{2.60}$$

and

$$X_n \diamond Y_n \quad \text{converges in } L^1(\mu) \quad \text{in } Z, \text{ say.} \tag{2.61}$$

Then we define $X \diamond Y = Z$.

The definition does not depend on the sequences $\{X_n\}, \{Y_n\}$. In fact, if (2.60)-(2.61) hold then

$$\mathcal{F}[Z](\phi) = e^{\frac{1}{2}\|\phi\|^2}\mathcal{F}[X](\phi)\mathcal{F}[Y](\phi) \quad ; \ \phi \in \mathcal{S}(\mathbf{R}^d) \tag{2.62}$$

where

$$\mathcal{F}[Z](\phi) = \int e^{i<\omega,\phi>}Z(\omega)d\mu(\omega) \tag{2.63}$$

is *the Fourier transform* of Z, and similarly with X and Y.

Note that if we choose $\phi = 0$ in (2.62) we get the following result of independent interest:

COROLLARY 2.10. Suppose $X, Y \in L^1(\mu)$ and that $X \diamond Y \in L^1(\mu)$ exists. Then

$$E[X \diamond Y] = E[X] \cdot E[Y]$$

For a further discussion of equation (2.59) we refer to [LØU 3].

§3. Discrete Wick calculus.

It is possible to develop a discrete version of Wick calculus which relates to Bernoulli random walks more or less the same way as the continuous theory relates to Brownian motion. Discrete Wick calculus has close connection to Meyer's "toy Fock space" version of quantum probability [M1], [M2] and it has much of the same purpose - to serve as a laboratory where one can observe the basic algebraic and probabilistic ideas without being disturbed by the heavy analytical machinery needed in the continuous theory. In this section we shall sketch briefly the basic ideas of discrete Wick calculus and indicate the connections to the continuous theory; more detailed information can be found in [HLØU].

Fix a finite set T, and let

$$\Omega = \{\omega | \omega : T \to \{-1, 1\}\}$$

be the space of all Bernoulli trials over T. By *toy Fock space* over T we shall simply mean the space $L^2(\Omega, P)$, where P is the uniform probability on Ω; i.e. the set of all functions $X : \Omega \to \mathbf{C}$ with the norm

$$\|X\| = (\sum |X(\omega)|^2 P(\omega))^{\frac{1}{2}}$$

For each $A \subset T$, define a function $\chi_A : \Omega \to \mathbf{C}$ by

$$\chi_A(\omega) = \begin{cases} \prod\limits_{a \in A} \omega(a) & \text{if } A \neq \emptyset \\ 1 & \text{if } A = \emptyset \end{cases}$$

Clearly, $\{\chi_A : A \subset T\}$ is an orthonormal set in $L^2(\Omega, P)$, and since its cardinality equals the dimension of $L^2(\Omega, P)$, it must be a basis. Hence any function $X \in L^2(\Omega, P)$ can be written uniquely as a sum

$$X = \sum_{A \subset T} X(A)\chi_A,$$

where each $X(A)$ is a complex number. We shall refer to this as the *Walsh decomposition* of X, and - as we shall see later - it is a close relative of Wiener-Itô chaos.

We are now ready to introduce the discrete Wick product on $L^2(\Omega, P)$. Intuitively, the idea is that we want an algebraic operation $\diamond$ such that

$$\chi_A \diamond \chi_B = \begin{cases} \chi_{A \cup B} & \text{if } A \cap B = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

and if we extend by ordinary algebraic rules, we are led to the following definition:

DEFINITION 3.1. If $X = \sum X_A \chi_A$ and $Y = \sum Y_B \chi_B$ are two elements of $L^2(\Omega, P)$, their *Wick product* $Z = X \diamond Y$ is defined by $Z = \sum Z_C \chi_C$, where

$$Z(C) = \sum \{X(A)Y(B) | A \cup B = C, A \cap B = \emptyset\}$$

REMARK 3.2. If U is the ordinary product of X and Y, then the Walsh components of U are

$$U(C) = \sum \{X(A)Y(B) | A \Delta B = C\},$$

and hence in one respect the Wick product is simpler than the ordinary product; in order to compute $Z(C)$ we only need to know the Walsh components $X(A)$ and $Y(B)$ for *subsets of* C, while $U(C)$ depends on *all* Walsh components of X and Y.

Let us make a useful but trivial observation:

LEMMA 3.3. $(L^2(\Omega, P), +, \diamond)$ is a commutative ring with unit element $\chi_\emptyset = 1$.

With a well-defined Wick product it is easy to define Wick polynomials. If $p(z_1, \cdots, z_k) = \sum_{|\alpha|=0}^{m} c_\alpha z^\alpha$ is a complex polynomial in k variables (where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_k)$ runs over multi-indices), and $X_1, \cdots, X_k \in L^2(\Omega, P)$, we define

$$p^\diamond(X_1, \cdots, X_k) = \sum_{|\alpha|=0}^{m} c_\alpha X_1^{\diamond\alpha_1} \diamond \cdots \diamond X_k^{\diamond\alpha_k},$$

where, of course, $X^{\diamond n} = X \diamond \cdots \diamond X$ (n times) for every positive integer n, and $X^{\diamond 0} = 1$. Power series can be treated similarly as long as we are a little careful with convergence problems (see [HLØU] for the details).

To state the main result of discrete Wick calculus, we need the following notation. If $\alpha = (\alpha_1, \cdots, \alpha_k)$ is a multi-index and A is a subset of T, then $\mathcal{P}_\alpha(A)$ is the family of all sequences of sets

$$< \{A_1^{(1)}, A_2^{(1)}, \cdots, A_{\alpha_1}^{(1)}\}, \{A_1^{(2)}, \cdots, A_{\alpha_2}^{(2)}\}, \cdots, \{A_1^{(k)}, \cdots, A_{\alpha_k}^{(k)}\} >,$$

such that the sets $A_j^{(i)}$ are non-empty and $\bigcup_{i,j} A_j^{(i)} = A$. If $\alpha_m = 0$, we just let $\{A_1^{(m)}, \cdots, A_{\alpha_m}^{(m)}\}$ be the empty set. We shall write $< A_j^{(i)} >$ as an abbreviation of $< \{A_1^{(1)}, \cdots, A_{\alpha_1}^{(1)}\}, \cdots, \{A_1^{(k)}, \cdots, A_{\alpha_k}^{(k)}\} >$.

THEOREM 3.4. Assume that $F : \mathbf{C}^N \times C^k \to \mathbf{C}^N$ is an analytic function, and let $Y_1, \cdots, Y_k : \Omega \to \mathbf{C}$ be random variables with expectations $y_1, \cdots, y_k$, respectively. Assume also that the Jacobian matrix

$$JF(x_1, \cdots, x_N; y_1, \cdots, y_k) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial F_N}{\partial x_1} & \frac{\partial F_N}{\partial x_2} & \cdots & \frac{\partial F_N}{\partial x_N} \end{pmatrix}$$

is nonsingular. Then there is a one-to-one correspondence between solutions $x_1, \cdots, x_N$ of the deterministic equation

$$F(x_1, \cdots, x_N, y_1, \cdots, y_k) = 0$$

and solutions $X_1, \cdots, X_N$ of the stochastic equation

$$F^\diamond(X_1, \cdots, X_N, Y_1, \cdots, Y_k) = 0.$$

Given a solution $x = (x_1, \cdots, x_N)$ of the deterministic equation, the corresponding solution $X = (X_1, \cdots, X_N)$ of the stochastic equation is given by the following hierarchy of equations:

$$X(\emptyset) = x,$$

and if $X(B)$ has been determined for all proper subsets B of A, then $X(A)$ is given by

$$X(A) = -JF(x_1, \cdots, x_N; y_1, \cdots, y_k)^{-1}[\sum_{l=1}^{k} \frac{\partial F}{\partial y_l}(x_1, \cdots, y_k)Y_l(A)+$$

$$+ \sum_{\substack{\alpha \\ |\alpha|>1}} \sum_{<\{A_j^i\}>\in\mathcal{P}_\alpha(A)} D_\alpha F(x_1, \cdots, y_k)X_1(A_1^{(1)}) \cdots Y_k(A_{\alpha_{N+k}}^{(N+k)})].$$

□

The proof of this theorem can be found in [HLØU]. Although the statement may seem quite complicated and confusing in the present setting, the main message is simple; if we can solve the nonlinear, deterministic equation

$$F(x_1, \cdots, x_N, y_1, \cdots, y_k) = 0,$$

we can also solve the nonlinear, stochastic equation

$$F^\diamond(X_1, \cdots, X_N, Y_1, \cdots, Y_k) = 0,$$

by solving an additional hierarchy of very simple linear equations (but, it should be admitted, this hierarchy is very large). Note that the form of our equation is more than general enough to cover all kinds of finite difference equations (a few examples are worked out in [HLØU]).

Let us end this section by pointing out the close formal relationship between discrete and continuous Wick calculus. For simplicity, we shall only compare the discrete theory with the one-dimensional version of the continuous theory. Assume that our underlying space T is a timeline

$$T = \{-T_0, -T_0 + \Delta t, \cdots, T_0 - \Delta t, T_0\}$$

where we think of Δt as a small time-increment. Let $B : \Omega \times T \to \mathbf{R}$ be the random walk

$$B(\omega, t) = \sum_{s<t} \omega(s)\sqrt{\Delta t},$$

and note that $\Delta B(\omega, t) \equiv B(\omega, t + \Delta t) - B(\omega, t) = \omega(t)\sqrt{\Delta t}$.

Given an $X \in L^2(\Omega, P)$, we now define a function $X_n : T^n \to \mathbf{C}$ for each $n \in \mathbf{N}$ by

$$X_n(t_1, \cdots, t_n) = \begin{cases} \frac{1}{(\Delta t)^{n/2} n!} X(\{t_1, \cdots, t_n\}) & \text{if } t_1, \cdots, t_n \text{ are distinct} \\ 0 & \text{otherwise} \end{cases}$$

where $X(\{t_1, \cdots, t_n\})$ is the Walsh component of X. Note that X_n is symmetric by definition, and that by rewriting the Walsh decomposition, we get

$$
\begin{aligned}
X &= \sum_n \sum_{t_1 < t_2 < \cdots < t_n} n! X_n(t_1, \cdots, t_n) \Delta t^{n/2} \chi_{\{t_1, \cdots, t_n\}} \\
&= \sum_n \sum_{(t_1, \cdots, t_n) \in T^n} X_n(t_1, \cdots, t_n) \Delta B(t_1) \cdots \Delta B(t_n)
\end{aligned}
$$

which is the discrete Wiener-Itô decomposition.

Returning to the Wick product, we now try to compute $(X \diamond Y)_n(t_1, \cdots, t_n)$. We get

$$
\begin{aligned}
(X \diamond Y)_n(t_1, \cdots, t_n) &= \frac{1}{(\Delta t)^{n/2} n!} (X \diamond Y)\{t_1, \cdots, t_n\} \\
&= \frac{1}{(\Delta t)^{n/2} n!} \sum_{k=0}^{n} \sum_{t_{i_1}, \cdots, t_{i_k}} X\{t_{i_1}, \cdots, t_{i_k}\} \cdot Y\{t_{j_1}, \cdots, t_{j_{n-k}}\}
\end{aligned}
$$

where $t_{i_1}, \cdots, t_{i_k}$ is a k-element subset of $\{t_1, \cdots, t_n\}$ and $t_{j_1}, \cdots, t_{j_{n-k}}$ are the remaining elements in $\{t_1, \cdots, t_n\}$. Continuing, we get

$$
\begin{aligned}
&(X \diamond Y)_n(t_1, \cdots, t_n) = \\
&= \frac{1}{(\Delta t)^{n/2} n!} \sum_{k=0}^{n} \sum_{t_{i_1} \cdots t_{i_k}} (\Delta t)^{k/2} k! X_k(t_{i_1}, \cdots, t_{i_k}) (\Delta t)^{(n-k)/2} (n-k)! Y_{n-k}(t_{j_1}, \cdots, t_{j_{n-k}}) \\
&= \sum_{k=0}^{n} \binom{n}{k}^{-1} \sum_{t_{i_1} \cdots t_{i_k}} X_k(t_{i_1}, \cdots, t_{i_k}) Y_{n-k}(t_{j_1}, \cdots, t_{j_{n-k}}) \\
&= \sum_{k=0}^{n} (X_k \hat{\otimes} Y_{n-k})(t_1, \cdots, t_n),
\end{aligned}
$$

where the last step uses that since X_k and Y_{n-k} already are symmetric, it suffices to symmetrize over all possible ways of distributing the t's between X_k and Y_{n-k}. It follows that

$$
X \diamond Y = \sum_{n,m} \sum_{(t_1, \cdots, t_{n+m})} (X_n \hat{\otimes} Y_m)(t_1, \cdots, t_{n+m}) \Delta B(t_1) \cdots \Delta B(t_{n+m})
$$

and hence the discrete and the continuous Wick products take on exactly the same form.

§4. Other characterizations of the Wick product.

4a) The $\mathcal{S}$-TRANSFORM AND THE HERMITE TRANSFORM

If $F \in (\mathcal{S})^*$ then the *$\mathcal{T}$-transform* of $F, \mathcal{T}F$, is a map from $\mathcal{S}(\mathbf{R}^d)$ into the complex plane $\mathbf{C}$ defined by

$$\mathcal{T}F(\phi) = < F, \exp(i < \cdot, \phi >) > \tag{4.1}$$

(It can be proved that the function $\omega \to \exp(i < \omega, \phi >); \omega \in \mathcal{S}'(\mathbf{R}^d)$ belongs to $(\mathcal{S})$, so (4.1) is well-defined).

Note that if $F \in L^2(\mu)$ then

$$\mathcal{T}F(\phi) = \int_{\mathcal{S}'(\mathbf{R}^d)} \exp(i < \omega, \phi >) F(\omega) d\mu(\omega) \tag{4.2}$$

so $\mathcal{T}$ coincides with the Fourier transform $\mathcal{F}$ in this case.

The *$\mathcal{S}$-transform* of $F, \mathcal{S}F$, is a map from $\mathcal{S}(\mathbf{R}^d)$ into $\mathbf{C}$ defined by

$$\mathcal{S}F(\phi) = e^{\frac{1}{2}\|\phi\|^2} \mathcal{T}F(-i\phi), \tag{4.3}$$

where $\|\phi\|^2 = \int_{\mathbf{R}^d} |\phi|^2 dx$ as before.

In particular, if $F \in L^2(\mu)$ we get

$$\mathcal{S}F(\phi) = e^{\frac{1}{2}\|\phi\|^2} \int_{\mathbf{R}^d} \exp(< \omega, \phi >) F(\omega) d\mu(\omega) \tag{4.4}$$

The *Hermite transform* of $F, \mathcal{H}F$ or $\tilde{F}$, is a map from the space $\mathbf{C}_0^{\mathbf{N}}$ of all finite sequences of complex numbers into $\mathbf{C}$ defined by

$$\mathcal{H}F(z_1, z_2, \cdots) = \tilde{F}(z_1, z_2, \cdots) = \mathcal{S}F(z_1 e_1 + z_2 e_2 + \cdots) \; ; \; (z_1, z_2, \cdots) \in \mathbf{C}_0^{\mathbf{N}} \tag{4.5}$$

where as before $\{e_n\}$ is the given orthonormal basis of $L^2(\mathbf{R}^d)$. Equivalently (see [LØU 1, Th. 5.7]), if $F \in (\mathcal{S})^*$ has the expansion

$$F(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega)$$

then, using multi-index notation $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots$ if $z = (z_1, z_2, \cdots)$ and $\alpha = (\alpha_1, \alpha_2, \cdots)$, we have

$$\mathcal{H}F(z) = \sum_\alpha c_\alpha z^\alpha \quad ; \; z \in \mathbf{C}_0^{\mathbf{N}} \tag{4.6}$$

It can be shown that this series converges and represents an analytic function of $z \in \mathbf{C}_0^{\mathbf{N}}$, for all $F \in (\mathcal{S})^*$. (See [HKPS]).

The $\mathcal{S}$-transform was first introduced in [KT]. The $\mathcal{H}$-transform and its inverse (see below) was first used in [LØU1], [LØU2] (on $L^2(\mu)$) as a tool to solve stochastic differential equations.

One important property of the $\mathcal{H}$-transform is the following, which follows directly from (4.6) and the definition of the Wick product:

THEOREM 4.1 a) Let $F, G \in (\mathcal{S})^*$. Then

$$\mathcal{H}(F \diamond G)(z) = \mathcal{H}F(z) \cdot \mathcal{H}G(z) \quad ; \quad z \in \mathbf{C}_0^{\mathbf{N}}$$

where the product on the right hand side is the usual complex product.

b) Alternatively, using $\mathcal{S}$-transforms we get

$$\mathcal{S}(F \diamond G)(\phi) = \mathcal{S}F(\phi) \cdot \mathcal{S}G(\phi) \quad ; \phi \in \mathcal{S}(\mathbf{R}^d)$$

EXAMPLE 4.2. We can use the Hermite transform to prove that there does not exist an L^p function process with $p > 1$ which solves the 1-dimensional pressure equation (2.59) for fluid flow in a porous medium:

Suppose $X(\phi, x, \omega)$ is an L^p functional process satisfying the equation

$$\text{(4.7)} \qquad \begin{cases} (K(x) \diamond X'(x))' = -a \quad ; \quad x \in (0,1) \\ \qquad X(0) = X(1) = 0 \end{cases}$$

where $a = \int_{\mathbf{R}} \phi(y)dy$, $\phi \in \mathcal{S}(\mathbf{R})$ is fixed. As before $K(x,\omega) = \operatorname{Exp} W_{\phi_x}(\omega)$ (see (2.57)).

Applying the Hermite transform on this equation we get

$$\begin{cases} (\tilde{K} \cdot \tilde{X}'(x))' = -a \quad ; x \in (0,1) \\ \qquad \tilde{X}(0) = \tilde{X}(1) = 0 \end{cases}$$

The solution of this equation is

$$\tilde{X}(x) = \int_0^x (-at + A) \exp(-\tilde{W}_{\phi_t})dt$$

where A is the constant determined by the requirement $X(1) = 0$, i.e.

$$A = A(z) = a \int_0^1 t \exp(-\tilde{W}_{\phi_t})dt \cdot (\int_0^1 \exp(-\tilde{W}_{\phi_t})dt)^{-1}$$

However, as a function of $z = (z_1, z_2, \cdots)$ we see that $A(z)$ is not *analytic* because it has poles at the zeroes of the function

$$B(z) = \int_0^1 \exp(-\tilde{W}_{\phi_t}(z))dt$$

Thus $\tilde{X}(z)$ is not analytic. Therefore X is not even in $(\mathcal{S})^*$ and therefore not in $L^p(\mu)$ for $p > 1$.

This shows the need for including the L^1 functional processes in the general study of stochastic differential equations, as pointed out in §2 e.

We end this subsection by giving the explicit form of *the inverse of the $\mathcal{H}$-transform:*

Suppose $X(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega) \in L^2(\mu)$ has the Hermite transform

$$\tilde{X}(z) = \sum_\alpha c_\alpha z^\alpha \quad ; \; z = (z_1, z_2, \cdots) \in \mathbf{C}_0^{\mathbf{N}}$$

Let λ be the probability measure on $\mathbf{R}^{\mathbf{N}}$ defined by

$$\int_{\mathbf{R}^{\mathbf{N}}} f(y_1, \cdots, y_n)d\lambda(y) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} f(y)e^{-\frac{1}{2}|y|^2}dy$$

if f is a bounded measurable function of $y \in \mathbf{R}^{\mathbf{N}}$ only depending on the first n coordinates $y_1, \cdots, y_n$ of y. In other words, λ is the infinite product of the normalized Gaussian measure in $\mathbf{R}$. Then we obtain X from $\tilde{X}(z)$ by

$$X(\omega) = \lim_{n,k\to\infty} \int_{\mathbf{R}^{\mathbf{N}}} \tilde{X}^{(n,k)}(\theta + iy)d\lambda(y) \quad (\text{limit in } L^2(\mu)) \tag{4.8}$$

where

$$\theta + iy = (\theta_1 + iy_1, \theta_2 + iy_2, \cdots) \quad (\text{with } \theta_k = \int_{\mathbf{R}^d} e_k dB$$

as in (2.11)) and

$$\tilde{X}^{(n,k)}(z) = \sum_{\alpha \in J_{n,k}} c_\alpha z^\alpha \quad ; \; J_{n,k} = \{\alpha; |\alpha| \le n \quad \text{and } \alpha_j = 0 \quad \text{for } j > k\}$$

is the doubly truncated series for $\tilde{X}(z)$.

To prove (4.8) we combine (2.11) with the following well known formula for the Hermite polynomials:

$$h_n(x) = \int_{\mathbf{R}} (x + iy)^n e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}}$$

EXAMPLE. If we choose e.g. $\phi = e$, then $K_\phi = \text{Exp } W_\phi = \exp(W_\phi - \frac{1}{2})$ has the $\mathcal{H}$-transform $\tilde{K}_\phi(z) = \exp \tilde{W}_\phi(z) = \exp z_1$. We can now verify directly that

$$\int_{\mathbf{R}^N} \tilde{K}_\phi(\theta + iy) d\lambda(y) = \int_{\mathbf{R}^N} \exp(\theta_1 + iy_1) d\lambda(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \exp(\theta_1 + iy_1 - \frac{1}{2}|y_1|^2) dy_1$$
$$= \exp(\theta_1 - \frac{1}{2}) = \text{Exp } W_\phi = K_\phi,$$

by using the well-known formula

$$\int_{\mathbf{R}} e^{i\alpha x - \varepsilon x^2} dx = \sqrt{\frac{\pi}{\varepsilon}} \exp(-\frac{\alpha^2}{4\varepsilon}) \ ; \ \varepsilon > 0, \ \alpha \in \mathbf{R}.$$

4b) CONNECTION WITH CONVOLUTION

If $f \in (\mathcal{S})$ and $\eta \in \mathcal{S}'(\mathbf{R}^d)$ we define the η-shift of $f, \tau_\eta f$, as follows:

$$\tau_\eta f(\omega) = f(\omega + \eta) \quad \omega \in \mathcal{S}'(\mathbf{R}^d)$$

The following results are known [HKPS]:

(i) For each $\eta \in \mathcal{S}'(\mathbf{R}^d)$ the map

$$f \to \tau_\eta f$$

is continuous from $(\mathcal{S})$ into $(\mathcal{S})$.

(ii) For each $f \in (\mathcal{S})$ the map

$$\eta \to \tau_\eta f(\cdot)$$

is continuous from $\mathcal{S}'(\mathbf{R}^d)$ into $(\mathcal{S})$.

Moreover, from [Z2] we have

THEOREM 4.3 For $f \in (\mathcal{S}), F \in (\mathcal{S})^*$ define

$$\Gamma(f, F) : \mathcal{S}'(\mathbf{R}^d) \to \mathbf{R}$$

by

$$\Gamma(f, F)(\eta) = < F, \tau_\eta f > \quad ; \ \eta \in \mathcal{S}'(\mathbf{R}^d).$$

Then Γ is a continuous map from $(\mathcal{S}) \times (\mathcal{S})^*$ into $(\mathcal{S})$. In particular, the function

$$\Gamma(f, F)(\cdot) = < F, \tau_\cdot f > \quad \text{is in } (\mathcal{S}).$$

This allows us to make the following definition:

DEFINITION 4.4 Let $F, G \in (\mathcal{S})^*$. Then we define the *convolution* of F and G, $F * G$, by

$$(4.9) \qquad \begin{aligned} < F * G, f > &=< F, < G, \tau.f >> \\ &=< G, < F, \tau.f >> \quad ; \ f \in (\mathcal{S}). \end{aligned}$$

Remark. The identity in (4.9) can be proved by using the $\mathcal{S}$-transform:

Define $Z(f) =< F, < G, \tau.f >>; f \in (\mathcal{S})$. Then, for $\phi \in \mathcal{S}(\mathbf{R}^d)$,

$$\begin{aligned} \mathcal{S}Z(\phi) &=< Z, \exp < \cdot, \phi >> e^{\frac{1}{2}\|\phi\|^2} \\ &=< F, < G, \tau.(\exp < \cdot, \phi >) >> e^{\frac{1}{2}\|\phi\|^2} \\ &=< F, < G, \exp < \cdot, \phi >> \exp < \cdot, \phi >> \cdot e^{\frac{1}{2}\|\phi\|^2} \\ &=< F, \exp < \cdot, \phi >> \cdot < G, \exp < \cdot, \phi >> \cdot e^{\frac{1}{2}\|\phi\|^2} \\ &= \mathcal{S}F(\phi) \cdot \mathcal{S}G(\phi) \cdot e^{-\frac{1}{2}\|\phi\|^2} \end{aligned}$$

which shows that we may interchange F and G in (4.9).

Note that this calculation shows that

$$(4.10) \qquad \mathcal{S}(F * G)(\phi) = \mathcal{S}F(\phi) \cdot \mathcal{S}G(\phi) \cdot \exp(-\frac{1}{2}\|\phi\|^2)$$

and hence, if also $H \in (\mathcal{S})^*$,

$$(4.11) \qquad \mathcal{S}(F * G * H)(\phi) = \mathcal{S}F(\phi) \cdot \mathcal{S}G(\phi) \cdot \mathcal{S}H(\phi) \cdot \exp(-\|\phi\|^2)$$

According to the characterization theorem (Th. 4.40 in [HKPS]) there exist $J, K \in (\mathcal{S})^*$ such that, for $\phi \in \mathcal{S}(\mathbf{R}^d)$,

$$(4.12) \qquad \mathcal{S}J(\phi) = \exp(-\frac{1}{2}\|\phi\|^2) \quad \text{and} \quad \mathcal{S}K(\phi) = \exp(\|\phi\|^2)$$

So if we combine Theorem 4.1 b) with (4.10) and (4.11), respectively, we get the following result [K]:

COROLLARY 4.5. Let $F, G \in (\mathcal{S})^*$. Then

$$(4.13) \qquad F * G = F \diamond G \diamond J$$

and

$$(4.14) \qquad F \diamond G = F * G * K$$

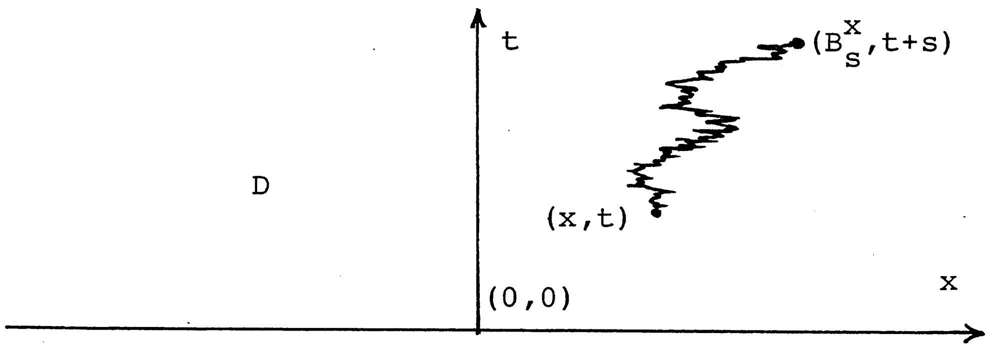
Remark. In [K] (4.13) is used a *definition* of the convolution $*$ and the identity (4.14) is proved from this. Thus Corollary 4.5 shows that our definition (4.9) is equivalent to the definition in [K].

4c) THE BACKWARD HEAT EQUATION

There is a simple and striking representation of the Wick product in terms of the backward (or, more precisely, the ill-posed) heat equation problem:

Given a function $f(x); x \in \mathbf{R}$ find $F(x,t)$ such that

$$(4.15) \qquad \begin{cases} \frac{\partial}{\partial t}F(x,t) + \frac{1}{2} \cdot \frac{\partial^2}{\partial x^2}F(x,t) = 0 \quad \text{in } D = \{(x,t); t > 0\} \\ \lim\limits_{t \to 0^+} F(x,t) = f(x); x \in \mathbf{R} \end{cases}$$



Remark. Note that if the domain $D = \{(x,t); t > 0\}$ is changed to $D' = \{(x,t); t < 0\}$ then the solution of the corresponding problem (4.15)' is given (under weak conditions on f) by

$$F(x,t) = E^{x,t}[f(B_\tau)],$$

where $E^{x,t}$ is the expectation with respect to the law $P^{x,t}$ of space-time Brownian motion $X_s = (B_s^x, t+s); s \geq 0$, starting at (x,t), and τ is the first exit time from D' of X_s. However, in the case considered in (4.15) one can only expect a solution to exist for more special choices of f.

The connection between Wick products and problem (4.15) seems to be a part of the folklore among experts in quantum field theory, although it seems hard to find it in the literature. Recently a more thorough investigation has been done [N]. We will here only sketch the main ideas and refer to [N] for proofs and more details.

First we establish that if $f(x) = x^n$, then the solution of (4.15) is given by

$$(4.16) \qquad F(x,t) = H_n(x,t) := t^{n/2} h_n\left(\frac{x}{\sqrt{t}}\right)$$

where $\{h_n\}$ are the Hermite polynomials defined in (1.10). $H_n(x, \sigma^2)$ is sometimes called the *n'th Hermite polynomial with parameter* $\sigma > 0$. To prove that $H_n(x,t)$ solves (4.15) we first note that by (2.47) we have

$$(4.17) \qquad \sum_{n=0}^{\infty} \frac{u^n}{n!} H_n(x,t) = \exp(ux - \frac{1}{2}u^2 t); u \in \mathbf{R}$$

From (4.17) we deduce that

$$\frac{\partial}{\partial x} H_n(x,t) = nH_{n-1}(x,t) \tag{4.18}$$

and

$$\frac{\partial}{\partial t} H_n(x,t) = -\frac{n(n-1)}{2} H_{n-2}(x,t). \tag{4.19}$$

It follows that $F(x,t) = H_n(x,t)$ satisfies the backward heat equation. Moreover, we see that

$$\lim_{t\to 0} H_n(x,t) = x^n \tag{4.20}$$

We conclude that to the given boundary value

$$f(x) = x^n$$

there corresponds the solution

$$F(x,t) = H_n(x,t)$$

More generally, we are interested in the correspondence $\mathcal{K}$ given by

$$f \to F = \mathcal{K}f$$

between boundary values f and corresponding solutions F of (4.15). The correspondence is clearly linear, so taking limits of sums of the polynomials above we obtain the following list of examples:

$f(x)$	$F(x,t) = \mathcal{K}f(x,t)$
x^n	$H_n(x,t)$
$e^{cx} = \sum_{n=0}^{\infty} \frac{1}{n!}(cx)^n$	$\sum_{n=0}^{\infty} \frac{c^n}{n!} H_n(x,t) = \exp(cx - \frac{1}{2}c^2 t)$
$\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$	$\frac{1}{2}(\exp(ix + \frac{1}{2}t) + \exp(-ix + \frac{1}{2}t)) = e^{\frac{1}{2}t}\cos x$
$\sin x$	$e^{\frac{1}{2}t}\sin x$
$\sum_{n=0}^{\infty} a_n x^n$ (with suitable convergence conditions)	$\sum_{n=0}^{\infty} a_n H_n(x,t)$

How does this correspondence behave under products? The table above indicates the answer:

If $f(x)=\sum\limits_{n=0}^{\infty} a_n x^n, g(x)=\sum\limits_{n=0}^{\infty} b_m x^m$, then $fg(x)=\sum\limits_{k=0}^{\infty} c_k x^k$ where $c_k=\sum\limits_{n+m=k} a_n b_m$. Therefore, if

$$\mathcal{K}f=F=\sum_{n=0}^{\infty} a_n H_n(x,t), \mathcal{K}g=G=\sum_{m=0}^{\infty} b_m H_m(x,t)$$

then

$$\mathcal{K}(fg)=\sum_{k=0}^{\infty} c_k H_k(x,t),$$

which is strikingly similar to the formula (2.31) for the Wick product based on Hermite expansions in the white noise probability space. In view of this the following definition is natural.

DEFINITION 4.6 Suppose $F=\mathcal{K}f, G=\mathcal{K}g$ are solutions of (4.15) with boundary values f,g. Then *the backward heat equation (b.h.e.) Wick product* of F and G, $F\tilde{\diamond}G$, is defined by

$$F\tilde{\diamond}G=\mathcal{K}(f\cdot g),$$

i.e. $F\tilde{\diamond}G$ is the solution of (4.15) with boundary values $f\cdot g$.

In this setting the b.h.e. Wick product is purely deterministic: It is just a product on a subset of the set of smooth functions on D. To see the explicit connection to the Wick product in stochastic analysis we use the following result, which may be deduced from (2.11):

$$n!\int\cdots\int_{0\le u_1\le\cdots\le u_n\le t}\Big(\Big(\int dB_{u_1}\Big)dB_{u_2}\Big)\cdots dB_{u_n}=t^{\frac{n}{2}}h_n\Big(\frac{B_t}{\sqrt{t}}\Big) \tag{4.21}$$

By (2.29) we can rewrite this to

$$B_t^{\diamond n}=H_n(B_t,t)=x^{\tilde{\diamond}n}|_{x=B_t} \tag{4.22}$$

More generally, if X_t is a (real) continuous semimartingale we define *the (semimartingale) Wick powers of* X_t by

$$X_t^{\hat{\diamond}n}=H_n(X_t,[X]_t), \tag{4.23}$$

where $[X]_t=\lim\limits_{\substack{\Delta t_j\to 0\\ t_j\le t}}\sum\limits_j (X_{t_{j+1}}-X_{t_j})^2$ is *the quadratic variation process* of X_t.

(Since this is not necessarily the same as the stochastic analysis Wick power applied to the random variable $X_t(\cdot)$ we use the notation $\hat{\diamond}$ here.) This gives the following definition of the (semimartingale) Wick product $(X\hat{\diamond}Y)_t$ of two semimartingales X_t, Y_t:

$$
(4.24)\qquad
\begin{aligned}
(X\hat{\diamond}Y)_t &= \frac{1}{4}\{(X_t+Y_t)^{\hat{\diamond}2} - (X_t-Y_t)^{\hat{\diamond}2}\} \\
&= \frac{1}{4}\{H_2(X_t+Y_t[X+Y]_t) - H_2(X_t-Y_t,[X-Y]_t)\} \\
&= \frac{1}{4}\{(X_t+Y_t)^2 - [X+Y]_t - ((X_t-Y_t)^2 - [X-Y]_t)\} \\
&= X_tY_t - \frac{1}{4}\{[X+Y]_t - [X-Y]_t\} = X_tY_t - [XY]_t,
\end{aligned}
$$

where $[XY]_t$ is the *quadratic covariation* of X and Y.

Finally we mention that the exponential in the semimartingale Wick sense gets the form

$$
(4.25)\qquad \hat{\operatorname{Exp}}(X_t) := \sum_{n=0}^{\infty} \frac{1}{n!} X_t^{\hat{\diamond}n} = \exp(X_t - \frac{1}{2}[X]_t)
$$

Fore more information we refer to [N].

§5. Properties of the Wick product.

The discussion in §2 shows that stochastic calculus based on Ito integrals (and Ito's differentiation rules) can be reformulated as Wick calculus where the usual products are replaced by Wick products and ordinary differentiation rules apply. This represents an obvious technical advantage and in addition it generalizes directly to the anticipative case (with Ito integrals replaced by Skorohod integrals). In view of this it is important to investigate further the properties of the Wick product.

5a) ALGEBRAIC PROPERTIES

Fix $Y \in (\mathcal{S})$ and consider the operator

$$X \to X \diamond Y \quad ; \; X \in L^2(\mu)$$

This is, for a generic choice of Y, an unbounded densely defined linear operator (See e.g. [M2]). As such the Wick product is not well behaved and must be treated with care. On the other hand it exhibits a number of nice algebraic properties, a few of which are listed below. Recall that if $F, G \in (\mathcal{S})^*$ then $F \diamond G \in (\mathcal{S})^*$ always. Similarly, $f, g \in (\mathcal{S}) \Rightarrow f \diamond g \in (\mathcal{S})$. If $X, Y \in L^1(\mu)$ however, we don't know if $X \diamond Y \in L^1(\mu)$ always exists.

PROPOSITION 5.1

(i) The Wick product is commutative, distributive and associative on $(\mathcal{S})^*$ and - when defined - on $L^1(\mu)$.

(ii) If $X, Y \in (\mathcal{S})^*$ (or $X, Y \in L^1(\mu)$ and $X \diamond Y \in L^1(\mu)$ exists) and $X \diamond Y = 0$, then either $X = 0$ or $Y = 0$.

(iii) If $X, Y \in L^1(\mu)$ and $X \diamond Y$ is defined in $L^1(\mu)$ then

$$\mathcal{F}[X \diamond Y](\phi) = e^{\frac{1}{2}\|\phi\|^2} \mathcal{F}[X](\phi) \cdot \mathcal{F}[Y](\phi) \quad \forall \phi \in \mathcal{S}(\mathbf{R}^d)$$

(See formula (2.62))

(iv) If $X, Y \in L^1(\mu)$ and $X \diamond Y \in L^1(\mu)$ is defined then

$$E[X \diamond Y] = E[X] \cdot E[Y] \qquad \text{(special case of (iii))}$$

(v) $F, G \in (\mathcal{S})^* \Rightarrow \mathcal{H}(F \diamond G)(z) = \mathcal{H}F(z) \cdot \mathcal{H}G(z) \quad ; \; z \in \mathbf{C}_0^{\mathbf{N}}$
and $\mathcal{S}(F \diamond G)(\phi) = \mathcal{S}F(\phi) \cdot \mathcal{S}G(\phi) \quad ; \; \phi \in \mathcal{S}(\mathbf{R}^d)$ (Theorem 4.1)

(vi) $F, G \in (\mathcal{S})^* \Rightarrow F * G = F \diamond G \diamond J$
and $F \diamond G = F * G * K$ (Corollary (4.5)

Proof. The only statement that is not proved already is (ii). This was first proved in [G]. In the $(\mathcal{S})^*$ setting we note that a short proof can be given using the Hermite transform:

If $X, Y \in (\mathcal{S})^*$ and $X \diamond Y = 0$ then

$$(5.1) \qquad 0 = \mathcal{H}(X \diamond Y) = \mathcal{H}X(z) \cdot \mathcal{H}Y(z);\ z \in \mathbf{C}_0^{\mathbf{N}}$$

Fix M and consider the analytic functions

$$\tilde{X}^{(M)}(z_1, \cdots, z_M) = \mathcal{H}X(z_1, \cdots, z_M, 0, \cdots) \quad ; \ (z_1, \cdots, z_M) \in \mathbf{C}^M$$
$$\tilde{Y}^{(M)}(z_1, \cdots, z_M) = \mathcal{H}Y(z_1, \cdots, z_M, 0, \cdots) \quad ; \ (z_1, \cdots, z_M) \in \mathbf{C}^M$$

By (5.1) we have $\tilde{X}^{(M)}\tilde{Y}^{(M)} \equiv 0$ on $\mathbf{C}^M$ and this is only possible if either $\tilde{X}^{(M)} \equiv 0$ or $\tilde{Y}^{(M)} \equiv 0$. Letting $M \to \infty$ we see that either $\tilde{X}^{(M)} \equiv 0$ for infinitely many M or $\tilde{Y}^{(M)} \equiv 0$ for infinitely many M. This gives that either $\mathcal{H}X \equiv 0$ or $\mathcal{H}Y \equiv 0$ and the conclusion follows.

We refer to [G] for a proof in the $L^1(\mu)$ case.

REMARKS

Property (iv) has the consequence that the average value $E[X(\phi, x, \cdot)]$ of the solution $X(\phi, x, \omega)$ of a stochastic differential equation involving Wick products coincides with the solution of the corresponding deterministic equation where all noise terms are replaced by their expected value. For example, the expected value $u(\phi, x) = E[X(\phi, x, \cdot)]$ of a solution X of equation (2.59) must satisfy the equation

$$\Delta_x u(\phi, x) = -(f * \phi)(x); \quad x \in D \subset \mathbf{R}^d$$

(where $\Delta_x = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator), since

$$E[K(\phi, x)] = E[\mathrm{Exp} W_{\phi_x}] = 1$$

It is natural to build analytic functions of white noise using Taylor expansions with powers replaced by Wick powers. One example is $\mathrm{Exp} W_\phi$, which was introduced in Example 2.7, formula (2.57). More generally, given $X \in L^1(\mu)$ we put

$$(5.2) \qquad \mathrm{Exp}[X] := \sum_{n=0}^{\infty} \frac{1}{n!} X^{\diamond n}$$

(when all the Wick powers of X exist and the series converges in $L^1(\mu)$). Here is a list of examples and their properties:

PROPOSITION 5.2

(i) $\mathrm{Exp}[W_\phi] := \sum_{n=0}^{\infty} \frac{1}{n!} W_\phi^{\diamond n} = \exp(W_\phi - \frac{1}{2}\|\phi\|^2)$

(ii) $\mathrm{Cos}[W_\phi] := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} W_\phi^{\diamond 2n} = \exp(\frac{1}{2}\|\phi\|^2) \cdot \cos W_\phi$

(iii) $\mathrm{Sin}[W_\phi] := \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} W_\phi^{\diamond(2n+1)} = \exp(\frac{1}{2}\|\phi\|^2) \cdot \sin W_\phi$

(iv) $\mathrm{Exp}[X] \diamond \mathrm{Exp}[Y] = \mathrm{Exp}[X+Y]$ (when defined)

(v) $\mathrm{Exp}[-\mathrm{Exp}[W_\phi]] = \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{n!} \exp(nW_\phi - \frac{n^2}{2}\|\phi\|^2)$

REMARKS. We have already proved (i) (see Example 2.7). Then (ii) and (iii) can be proved by using that

$$\mathrm{Cos}[W_\phi] = \frac{1}{2}(\mathrm{Exp}[iW_\phi] + \mathrm{Exp}[-iW_\phi])$$

and

$$\mathrm{Sin}[W_\phi] = \frac{1}{2i}(\mathrm{Exp}[iW_\phi] - \mathrm{Exp}[-iW_\phi])$$

(Note that $iW_\phi = W_{i\phi}$).

To prove (iv) we repeat the algebraic proof that $\exp x \cdot \exp y = \exp(x+y); x, y \in \mathbf{R}$, but with Wick products replacing ordinary products.

We establish (v) by applying the Exp-definition twice and noting that we have convergence in $L^1(\mu)$. Formula (v) shows that *in general $\mathrm{Exp}[X]$ need not be positive, a.s. - in fact not even bounded below.*

For example, if $y^2 = \frac{1}{2}x$ then it is easily seen that

$$f(x,y) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \exp(nx - n^2y^2) < -M$$

for $x > 2\ln(3+M)$.

We now discuss positivity in more detail.

DEFINITION 5.3
If $F \in (\mathcal{S})^*$ we say that F is *positive* (and write $F \geq 0$) if

$$< F, f > \geq 0 \quad \text{for all} \quad f \in (\mathcal{S}), f \geq 0. \tag{5.3}$$

In particular, if $X \in L^2(\mu)$ then $X \geq 0$ iff

$$< X, f > = E[X \cdot f] \geq 0 \quad \text{for all} \quad f \in (\mathcal{S}); f \geq 0$$

(see (2.23) and below). This occurs iff

$$X(\omega) \geq 0 \quad \text{for} \quad a.a.\omega$$

In view of this the following definition is natural:

DEFINITION 5.4 An L^p functional process $X(\phi, x, \omega)$ is called *positive* if

$$X(\phi, x, \omega) \geq 0 \quad \text{a.s.} \quad \forall \phi \in \mathcal{S}(\mathbf{R}^d), \forall x \in \mathbf{R}^d$$

It is known ([K]) that in general positivity is *not* preserved by Wick products. However, we do have the following result [LØU, Corollary 7.5]:

THEOREM 5.5.
Let

$$X(\phi, x, \omega) = \sum_\alpha a_\alpha(\phi, x) H_\alpha(\omega)$$

and

$$Y(\phi, x, \omega) = \sum_\beta b_\beta(\phi, x) H_\beta(\omega)$$

be L^2 functional processes which are *homogeneous*, in the sense that

$$a_\alpha(c\phi, x) = c^{|\alpha|} a_\alpha(\phi, x), \tag{5.4}$$

$$b_\beta(c\phi, x) = c^{|\beta|} b_\beta(\phi, x) \tag{5.5}$$

for all multi-indeces α, β and all $c \in \mathbf{R}$, $\phi \in \mathcal{S}(\mathbf{R}^d)$. Suppose X, Y are positive (as functional processes). Then

$$X \diamond Y$$

is also a positive functional process (when defined).

In general one might say that Wick multiplication relates to Hermite polynomials in the same way as ordinary multiplication relates to the polynomials x^n. It is therefore of interest to investigate the algebraic properties of Hermite polynomials. We mention one useful result, a direct proof of which can be found in [HLØUZ]:

PROPOSITION 5.6
If $a = (a_1, \cdots, a_N)$ with $\sum\limits_{j=1}^{N} a_j^2 = 1$, then

$$h_n\left(\sum_{j=1}^{N} a_j x_j\right) = \sum_\alpha \frac{n!}{\alpha!} a^\alpha h_\alpha(x), \tag{5.6}$$

the summation being taken over all $\alpha = (\alpha_1, \cdots, \alpha_N)$ such that $|\alpha| = n$. ($\alpha! = \alpha_1! \cdots \alpha_N!$ and $h_\alpha(x) = \prod\limits_{j=1}^{N} h_{\alpha_j}(x_j), x = (x_1, \cdots, x_N)$).

REMARK. It is of interest to note that the Wick calculus we have already established can be used to prove results about Hermite polynomials. For example, we can now give a short proof of Proposition 5.6 as follows:

Choose $c = (c_1, \cdots, c_N) \in \mathbf{R}^N$ and put $\phi = \sum_{j=1}^{N} c_j e_j$. Then by (2.33) we have

$$\|\phi\|^2 h_n\left(\frac{W_\phi}{\|\phi\|}\right) = W_\phi^{\diamond n} = \left(\sum_{j=1}^{N} c_j W_{e_j}\right)^{\diamond n} = \sum_\alpha \frac{n!}{\alpha!} c^\alpha W_{(\alpha)}^{\diamond \alpha} \tag{5.7}$$

where we have used the fact that the multinomial formula looks the same for Wick products as for ordinary products, and we have adopted the notation

$$W_{(\alpha)}^{\diamond \alpha} = W_{e_{\alpha_1}}^{\diamond \alpha_1} \diamond \cdots \diamond W_{e_{\alpha_N}}^{\diamond \alpha_N} \tag{5.8}$$

By (2.29) and (2.11) we have

$$W_{e_{\alpha_1}}^{\diamond \alpha_1} \diamond \cdots \diamond W_{e_{\alpha_N}}^{\diamond \alpha_N} = \prod_{j=1}^{N} h_{\alpha_j}(\theta_j) \tag{5.9}$$

Combining (5.7)-(5.9) we conclude that

$$\|\phi\|^2 h_n\left(\frac{W_\phi}{\|\phi\|}\right) = \sum_\alpha \frac{n!}{\alpha!} c^\alpha h_\alpha(\theta) \tag{5.10}$$

where $\theta = (\theta_1, \cdots, \theta_N)$. In particular, if $\sum_{j=1}^{N} c_j^2 = 1$ we get

$$h_n\left(\sum_{j=1}^{N} c_j \theta_j\right) = \sum_\alpha \frac{n!}{\alpha!} c^\alpha h_\alpha(\theta) \tag{5.11}$$

Since this holds a.s. and $\theta_1, \cdots, \theta_N$ are independent normal random variables it must hold for all possible values of $x_j = \theta_j(\omega)$ and (5.6) follows.

5b) ANALYTIC PROPERTIES AND ESTIMATES

Unfortunately very little is known regarding L^p-estimates and continuity of the Wick product. Here is a basic connection between X and $\tilde{X}$ [HLØUZ, Th. 4.2]:

THEOREM 5.7
Let $X \in L^2(\mu)$ and let $\tilde{X}^{(n,k)}(z)$ be the truncated Hermite transform of X (see §4a). Then

$$E[|X|^p] \le \liminf_{n,k\to\infty} \int\int |\tilde{X}^{(n,k)}(\xi + i\eta)|^p d\lambda(\xi) d\lambda(\eta)$$

for all $p \ge 1$.

COROLLARY 5.8
Let $X_n \in L^2(\mu)$ for all n and assume that $\{\tilde{X}_n\}_{n=1}^{\infty}$ is a convergent sequence in $L^p(\lambda \times \lambda)$ for some $p \geq 1$. Then $\{X_n\}$ converges in $L^p(\mu)$.

COROLLARY 5.9
Suppose $X \in L^2(\mu)$ satisfies the condition

$$\sum \frac{1}{n!} \|\tilde{X}^n\|_{L^p(\lambda\times\lambda)} < \infty$$

for some $p \geq 1$. Then

$$\operatorname{Exp}(X) \in L^p(\mu)$$

Unfortunately Theorem 5.7 and its Corollaries are too crude to be really useful: In many important cases $\tilde{X}$ is not even in $L^1(\lambda \times \lambda)$. Some stronger results can be found in [G].

For Wick powers of white noise there exists a sharp estimate due to Carlen & Kree [CK] (already mentioned in Example 2.7). Formulated in our setting the result is the following:

THEOREM 5.10
Let $\phi \in \mathcal{S}(\mathbf{R}^d)$ and $p \geq 1$. There exist constants $A_{p,n}, B_{p,n}$ such that

$$n! A_{p,n} \|\phi\|^n \leq E[|W_\phi^{\diamond n}|^p]^{1/p} \leq n! B_{p,n} \|\phi\|^n$$

and

$$\ln A_{p,n} \sim \ln B_{p,n} \sim -\frac{n}{2} \ln n \quad \text{as } n \to \infty$$

5c) IS THE WICK PRODUCT LOCAL?

Unlike the ordinary (pointwise) product $(X \cdot Y)(\omega) = X(\omega) \cdot Y(\omega)$ of two random variables X, Y on $\mathcal{S}'$, the value of the Wick product $X \diamond Y$ at $\omega_0 \in \mathcal{S}'$ is in general *not* determined by $X(\omega_0)$ and $Y(\omega_0)$ alone but on (possibly) all the values $\{X(\omega); \omega \in \mathcal{S}'\}$ and $\{Y(\omega); \omega \in \mathcal{S}'\}$. This is not so surprising in view of the definition

$$(X \diamond Y)(\omega) = \sum_\gamma \Big(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \Big) H_\gamma(\omega)$$

(if $X = \sum\limits_\alpha a_\alpha H_\alpha, Y = \sum\limits_\beta b_\beta H_\beta$) or in view of its connection to convolution (Cor. 4.5)

$$X \diamond Y = F * G * K$$

In fact, in general the Wick product is not even *local*, in the following sense:

DEFINITION 5.11
Let $\mathcal{M} \subset L^1(\mu)$. We say that the Wick product is *local* on $\mathcal{M}$ if for all $X, Y \in \mathcal{M}$, all $\omega_0 \in \mathcal{S}'$ and for all (non-empty) open neighboorhoods U of ω_0 (in the weak-star topology on $\mathcal{S}'$) the value of $(X \diamond Y)(\omega_0)$ is determined by the values of X and Y on U alone.

PROPOSITION 5.12
Let $A \neq \phi$ be an open subset of $\mathcal{S}'$ such that $B = (\overline{A})^c \neq \emptyset$ and $\mu(\partial A) = 0$, where ∂A is the boundary of A and $\overline{A}$ is the closure of A. Suppose $\mathcal{M}$ contains X_1, X_2, Y_1, Y_2 such that

$$X_1 = X_2 \quad \text{a.s. on } A, \quad Y_1 = Y_2 \quad \text{a.s. on } B$$

and

$$\mu(X_1 \neq X_2) > 0, \quad \mu(Y_1 \neq Y_2) > 0.$$

Then the Wick product is not local on $\mathcal{M}$.

Proof. Note that

$$(X_1 - X_2) \diamond (Y_1 - Y_2) = \begin{cases} 0 \diamond (Y_1 - Y_2) & \text{on } A \\ (X_1 - X_2) \diamond 0 & \text{on } B \end{cases}$$

Hence, if $\diamond$ is local on $\mathcal{M}$ then $(X_1 - X_2) \diamond (Y_1 - Y_2) = 0$ on $A \cup B$ and hence a.s. on $\mathcal{S}'$. So from Proposition 5.1 (ii) we conclude that either $X_1 = X_2$ a.s. or $Y_1 = Y_2$ a.s., which contradicts our assumptions.

In view of Proposition 5.12 we conclude that usually the Wick product is not local unless each of the factors X, Y themselves are locally determined. For more details we refer to [G].

§6. Some open problems.

There are many interesting and natural problems related to the Wick product. We list some of them below. Some of the problems may be hard to solve and some easy - or even known. We would appreciate hearing from those who have information about any of them.

PROBLEM 1. Do there exist $p, q \geq 1$ such that

$$X, Y \in L^p(\mu) \Rightarrow X \diamond Y \quad \text{is defined and in } L^q(\mu)?$$

(Probably not, but it would be useful to have a counterexample)

PROBLEM 2. Extend the definition of $\diamond$ from $L^1(\mu)$ to a class $\mathcal{W} \supset L^1(\mu)$ of measurable functions (on $\mathcal{S}'$) in such a way that $X \diamond Y \in \mathcal{W}$ for all $X, Y \in \mathcal{W}$.

PROBLEM 3. For what X does there exist Y such that $X \diamond Y = 1$?

PROBLEM 4. If $X, Y \in L^1(\mu)$ are *independent*, is

$$X \diamond Y = X \cdot Y?$$

REMARK. The conclusion can be shown to hold if e.g.

$$X = \sum_n \int f_n dB^{\otimes n} \in L^2(\mu) \quad \text{and } Y = \sum_m \int g_m dB^{\otimes m} \in L^2(\mu)$$

are *strongly independent* in the sense that for all n, m the coordinate projections of $\operatorname{supp} f_n \subset \mathbf{R}^n$ and $\operatorname{supp} g_m \subset \mathbf{R}^m$ are disjoint.

PROBLEM 5. Extend the Hermite transform to $L^1(\mu)$.

REMARK. Example 4.2 indicates that for such an extension of $\mathcal{H}$ to $L^1(\mu)$ we can no longer expect that $\mathcal{H}X(z)$ is always analytic on $\mathbf{C}_0^{\mathbf{N}}$: For some $X \in L^1(\mu)$ the function $\mathcal{H}X(z)$ may have poles.

PROBLEM 6. For $X \in L^1(\mu)$ and $f : \mathbf{C} \to \mathbf{C}$ analytic there is a natural *Wick version of* $f(X)$, denoted by $f^\diamond(X)$, defined as follows:

$$f^\diamond(X) = \sum_{n=0}^{\infty} a_n X^{\diamond n} \quad \text{if } f(z) = \sum_{n=0}^{\infty} a_n z^n$$

(assuming that all the Wick powers exist and that the series to the left converges). More generally, suppose $f : \mathbf{R} \to \mathbf{R}$ is a given function (not necessarily real analytic) when can one make sense of a Wick version $f^\diamond(X)$?

REMARK. This question is discussed in connection with stochastic differential equations in [LØU 2].

PROBLEM 7. If $X, Y \in L^2(\mu)$ and (X, Y) is identical in law with another pair (X', Y') with $X', Y' \in L^2(\mu)$, is $X \diamond Y$ identical in law with $X' \diamond Y'$ (when defined)?

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REFERENCES

[AP] J. Ash and J. Potthoff: Ito's lemma without non-anticipatory conditions. Probab.Th.Rel.Fields 88 (1991), 17-46.

[B] R. Buckdahn: Linear Skorohod stochastic differential equations. Probab.Th.Rel. Fields 90 (1991), 223-240.

[CK] E. Carlen and P. Kree: L^p estimates on iterated stochastic integrals. Manuscript 1991.

[DM] R.L. Dobrushin and R.A. Minlos: Polynomials in linear random functions. Russian Math.Surveys 32 (2) (1977), 71-127.

[GV] I.M. Gelfand and N.Y. Vilenkin: Generalized Functions, Vol.4: Applications of Harmonic Analysis. Academic Press 1964 (English translation).

[G] H. Gjessing: Some properties of the Wick product and the Hermite transform. Manuscript 1992.

[GJ] J. Glimm and A. Jaffe: Quantum Physics. (2^{nd} edition) Springer-Verlag 1987.

[H] T. Hida: Brownian Motion. Springer-Verlag 1980.

[HI] T. Hida and N. Ikeda: Analysis on Hilbert space with reproducing kernel arising from multiple Wiener integral. Proc. Fifth Berkeley Symp. Math.Stat.Probab. II, part 1 (1965), 117-143.

[HKPS] T. Hida, H.-H. Kuo, J. Potthoff and L. Streit: White Noise Analysis. Forthcoming book.

[HLØU] H. Holden, T. Lindstrøm, B. Øksendal and J. Ubøe: Discrete Wick calculus and stochastic functional equations. Preprint University of Oslo 1992.

[HLØUZ] H. Holden, T. Lindstrøm, B. Øksendal, J. Ubøe and T.-S. Zhang: Stochastic boundary value problems. A white noise functional approach. Preprint University of Oslo 1991.

[I] K. Ito: Multiple Wiener integral. J.Math.Soc. Japan 3 (1951), 157-169.

[K] H.-H. Kuo: Convolution and Fourier transform of Hida distributions. Manuscript 1991.

[KT] I. Kubo and S. Takenaka: Calculus on Gaussian white noise I. Proc. Japan Acad. 56 (1980), 376-380.

[LØU1] T. Lindstrøm, B. Øksendal and J. Ubøe: Stochastic differential equations involving positive noise. In M. Barlow and N. Bingham (editors): Stochastic Analysis. Cambridge Univ. Press 1991, 261-303.

[LØU2] T. Lindstrøm, B. Øksendal and J. Ubøe: Wick multiplication and Ito-Skorohod stochastic differential equations. To appear in S. Albeverio et al (editors): Ideas and Methods in Mathematics Analysis. Cambridge Univ. Press 1992.

[LØU3] T. Lindstrøm, B. Øksendal and J. Ubøe: Stochastic modelling of fluid flow in porous media. Preprint University of Oslo 1991.

[M1] P.A. Meyer: A finite approximation to Boson Fock space. In S. Albeverio et al (editors): Stochastic Processes in Classical and Quantum Systems. Springer-Verlag 1986, 405-410.

[M2] P.A. Meyer: Fock space and probability theory. In S. Albeverio et al (editors): Stochastic Processes - Mathematics and Physics. Springer LNM 1250 (1987), 160-170.

[MY] P.A. Meyer and J.A. Yan: Distributions sur l'espace de Wiener (suite). Sém. de Probabilités XXIII, P.A. Meyer and M. Yor (editors). Springer LNM 1372 (1989).

[N] T. Norberg: A note on solutions of the backward heat equation. Manuscript 1992.

[NZ] D. Nualart and M. Zakai: Generalized stochastic integrals and the Malliavin calculus. Probab.Th.Rel.Fields 73 (1986), 255-280.

[NP] D. Nualart and E. Pardoux: Stochastic calculus with anticipating integrands. Probab.Th.Rel.Fields 78 (1988), 535-581.

[ØZ] B. Øksendal and T.-S. Zhang: The stochastic Volterra equation. Manuscript 1992.

[P] J. Potthoff: White noise methods for stochastic partial differential equations. Manuscript 1991.

[S] B. Simon: The $P(\phi)_2$ Euclidean (Quantum) Field Theory. Princeton University Press 1974.

[Wa] J.B. Walsh: An introduction to stochastic partial differential equations. In R.

Carmona, H. Kesten and J.B. Walsh (editors): École d'Été de Probabilités de Saint-Flour XIV-1984. Springer LNM 1180 (1986), 265-437.

[Wi] G.C. Wick: The evaluation of the collinear matrix. Phys.Rev. 80 (1950), 268-272.

[Z1] T.-S. Zhang: Characterizations of white noise test functions and Hida distributions. Preprint University of Oslo 1991. (To appear in Stochastics.)

[Z2] T.-S. Zhang: On convolution of Hida distributions. Preprint University of Oslo 1992.